

AFFINIZATIONS AND R-MATRICES FOR QUIVER HECKE ALGEBRAS

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ABSTRACT. We introduce the notion of affinizations and R -matrices for arbitrary quiver Hecke algebras. It is shown that they enjoy similar properties to those for symmetric quiver Hecke algebras. We next define a duality datum \mathcal{D} and construct a tensor functor $\mathfrak{F}^{\mathcal{D}}: \text{Mod}_{\text{gr}}(R^{\mathcal{D}}) \rightarrow \text{Mod}_{\text{gr}}(R)$ between graded module categories of quiver Hecke algebras R and $R^{\mathcal{D}}$ arising from \mathcal{D} . The functor $\mathfrak{F}^{\mathcal{D}}$ sends finite-dimensional modules to finite-dimensional modules, and is exact when $R^{\mathcal{D}}$ is of finite type. It is proved that affinizations of real simple modules and their R -matrices give a duality datum. Moreover, the corresponding duality functor sends a simple module to a simple module or zero when $R^{\mathcal{D}}$ is of finite type. We give several examples of the functors $\mathfrak{F}^{\mathcal{D}}$ from the graded module category of the quiver Hecke algebra of type D_{ℓ} , C_{ℓ} , $B_{\ell-1}$, $A_{\ell-1}$ to that of type A_{ℓ} , A_{ℓ} , B_{ℓ} , B_{ℓ} , respectively.

INTRODUCTION

The *quiver Hecke algebras* (or *Khovanov-Lauda-Rouquier algebras*), introduced by Khovanov-Lauda ([13, 14]) and Rouquier ([18]) independently, are \mathbb{Z} -graded algebras which provide a categorification for the negative half of a quantum group. The algebras are a vast generalization of affine Hecke algebras of type A in the direction of categorification ([1, 18]), and they have special graded quotients, called *cyclotomic quiver Hecke algebras*, which categorify irreducible integrable highest weight modules ([4]). When the quiver Hecke algebras are *symmetric*, we can study them more deeply.

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- First of all, it is known that the upper global basis corresponds to the set of the isomorphism classes of simple modules over symmetric quiver Hecke algebras ([19, 20]).
- The KLR-type quantum affine Schur-Weyl duality functor was constructed in [5] using symmetric quiver Hecke algebras and R -matrices of quantum affine algebras. This functor has been studied in various types ([6, 7, 10]).

The notion of R -matrices for symmetric quiver Hecke algebras was introduced in [5]. The R -matrices are special homomorphisms defined by using intertwiners and affinizations. It turned out that the R -matrices have very good properties with *real* simple modules ([11]). They also had taken an important role as a main tool in studying a monoidal categorification of quantum cluster algebras ([8, 9]).

Let us explain the construction of R -matrices in [5] briefly. We assume that the quiver Hecke algebra R is symmetric. Let M be an R -module and M_z its affinization. The R -module M_z is isomorphic to $\mathbf{k}[z] \otimes_{\mathbf{k}} M$ as a \mathbf{k} -vector space. The actions of $e(\nu)$ and τ_i on M_z are the same as those on M , but the action of x_i on M_z is equal to the action x_i on M added by z (see (1.8)). For R -modules M and N , we next consider the homomorphism $R_{M_z, N_{z'}} \in \text{HOM}_R(M_z \circ N_{z'}, N_{z'} \circ M_z)$ given by using intertwiners (see (1.7)). Here HOM denotes the non-graded homomorphism space (see (1.5)). We set

$$R_{M_z, N_{z'}}^{\text{norm}} := (z' - z)^{-s} R_{M_z, N_{z'}}, \quad \mathbf{r}_{M, N} := R_{M_z, N_{z'}}^{\text{norm}}|_{z=z'=0},$$

where s is the order of zero of $R_{M_z, N_{z'}}$. Then the morphisms $R_{M_z, N_{z'}}^{\text{norm}}$ and $\mathbf{r}_{M, N}$ are non-zero, commute with the spectral parameters z , z' , and satisfy the braid relations. Here, in defining M_z and $\mathbf{r}_{M, N}$, we crucially use the fact that R is symmetric.

In this paper, we introduce and investigate the notion of affinizations and R -matrices for *arbitrary* quiver Hecke algebras, and construct a new duality functor between finitely generated graded module categories of quiver Hecke algebras. The affinizations defined in this paper generalize the affinizations M_z for symmetric quiver Hecke algebras. The root modules given in [2] are examples of affinizations.

We then define a tensor functor $\mathfrak{F}^{\mathcal{D}} : \text{Mod}_{\text{gr}}(R^{\mathcal{D}}) \rightarrow \text{Mod}_{\text{gr}}(R)$ between graded module categories of quiver Hecke algebras R and $R^{\mathcal{D}}$, which arises from a *duality datum* \mathcal{D} consisting of certain R -modules and their homomorphisms. This is inspired by the KLR-type quantum affine Schur-Weyl duality functor in [5]. The functor $\mathfrak{F}^{\mathcal{D}}$ sends finite-dimensional modules to finite-dimensional modules. It is exact when $R^{\mathcal{D}}$ is of finite type. We show that affinizations of real simple modules and their R -matrices give a duality datum. The corresponding duality functor sends a simple module to a simple module or zero when $R^{\mathcal{D}}$ is of finite type.

Here is a brief description of our work. Let $R(\beta)$ be an arbitrary quiver Hecke algebra. We define an affinization $(\mathbf{M}, z_{\mathbf{M}})$ of a simple $R(\beta)$ -module \bar{M} to be an $R(\beta)$ -module \mathbf{M} with a homogeneous endomorphism $z_{\mathbf{M}} \in \text{End}_R(\mathbf{M})$ and an isomorphism $\mathbf{M}/z_{\mathbf{M}}\mathbf{M} \simeq \bar{M}$ satisfying the conditions in Definition 2.2. However, we do not know if every simple module admits an affinization.

We then study the endomorphism rings of affinizations and the homomorphism spaces between convolution products of simple modules and their affinizations. For a non-zero R -module N , let s be the largest integer such that $R_{\mathbf{M},N}(\mathbf{M} \circ N) \subset z_{\mathbf{M}}^s N \circ \mathbf{M}$. We set

$$R_{\mathbf{M},N}^{\text{norm}} = z_{\mathbf{M}}^{-s} R_{\mathbf{M},N}: \mathbf{M} \circ N \rightarrow N \circ \mathbf{M},$$

and denote by $\mathbf{r}_{\bar{M},N}: \bar{M} \circ N \rightarrow N \circ \bar{M}$ the homomorphism induced by $R_{\mathbf{M},N}^{\text{norm}}$. By the definition $\mathbf{r}_{\bar{M},N}$ never vanishes. The R -matrix $\mathbf{r}_{\bar{M},N}$ has similar properties to R -matrices for symmetric quiver Hecke algebras (Proposition 2.10). Proposition 2.12 tells us that, if $(\mathbf{M}, z_{\mathbf{M}})$ and $(\mathbf{N}, z_{\mathbf{N}})$ are affinization of simple modules \bar{M} and \bar{N} and one of \bar{M} and \bar{N} is real (see (2.7)), then

- (i) $\text{HOM}_{R[z_{\mathbf{M}}, z_{\mathbf{N}}]}(\mathbf{M} \circ \mathbf{N}, \mathbf{M} \circ \mathbf{N}) = \mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}] \text{id}_{\mathbf{M} \circ \mathbf{N}}$,
- (ii) $\text{HOM}_{R[z_{\mathbf{M}}, z_{\mathbf{N}}]}(\mathbf{M} \circ \mathbf{N}, \mathbf{N} \circ \mathbf{M})$ is a free $\mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]$ -module of rank one.

Here, HOM denotes the space of non-graded homomorphisms (see (1.5)). We define $R_{\mathbf{M},\mathbf{N}}^{\text{norm}}$ as a generator of the $\mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]$ -module $\text{HOM}_{R[z_{\mathbf{M}}, z_{\mathbf{N}}]}(\mathbf{M} \circ \mathbf{N}, \mathbf{N} \circ \mathbf{M})$. Then $R_{\mathbf{M},\mathbf{N}}^{\text{norm}}$ commutes with $z_{\mathbf{M}}$ and $z_{\mathbf{N}}$ by the construction, and we prove that $R_{\mathbf{M},\mathbf{N}}^{\text{norm}}|_{z_{\mathbf{M}}=z_{\mathbf{N}}=0} \in \text{HOM}(\bar{M} \circ \bar{N}, \bar{N} \circ \bar{M})$ does not vanish and coincides with $\mathbf{r}_{\bar{M},\bar{N}}$ up to a constant multiple in Theorem 2.13.

We next define the duality datum $\mathcal{D} = \{\beta_j, M_j, \mathbf{z}_j, \mathbf{r}_j, \mathbf{R}_{j,k}\}_{j,k \in J}$ axiomatically. Here, J is a finite index set, and

$$\begin{aligned} M_j &\in \text{Mod}_{\text{gr}}(R(\beta_j)), & \mathbf{z}_j &\in \text{END}_{R(\beta_j)}(M_j), \\ \mathbf{r}_j &\in \text{END}_{R(2\beta_j)}(M_j \circ M_j), & \mathbf{R}_{j,k} &\in \text{HOM}_{R(\beta_j+\beta_k)}(M_j \circ M_k, M_k \circ M_j), \end{aligned}$$

satisfying certain conditions given in Definition 4.1. We construct the generalized Cartan matrix $\mathbf{A}^{\mathcal{D}}$ and the polynomial parameters $\mathcal{Q}_{i,j}^{\mathcal{D}}(u, v)$ out of the duality datum \mathcal{D} and consider the quiver Hecke algebra $R^{\mathcal{D}}$ corresponding to $\mathbf{A}^{\mathcal{D}}$ and $\mathcal{Q}_{i,j}^{\mathcal{D}}(u, v)$. For $\gamma \in \mathbf{Q}_+^{\mathcal{D}}$ with $m = \text{ht}(\gamma)$, we define

$$\Delta^{\mathcal{D}}(\gamma) := \bigoplus_{\mu \in J^{\gamma}} \Delta_{\mu}^{\mathcal{D}},$$

where

$$\Delta_{\mu}^{\mathcal{D}} := M_{\mu_1} \circ M_{\mu_2} \circ \cdots \circ M_{\mu_m} \text{ for } \mu = (\mu_1, \mu_2, \dots, \mu_m) \in J^{\gamma}.$$

It turns out that $\Delta^{\mathcal{D}}(\gamma)$ has an $(R, R^{\mathcal{D}})$ -bimodule structure (Theorem 4.2), and we obtain the duality functor $\mathfrak{F}^{\mathcal{D}}: \text{Mod}_{\text{gr}}(R^{\mathcal{D}}) \rightarrow \text{Mod}_{\text{gr}}(R)$ by tensoring $\Delta^{\mathcal{D}}(\gamma)$. Theorem 4.3 tells that $\mathfrak{F}^{\mathcal{D}}$ is a tensor functor and sends finite-dimensional modules to finite-dimensional modules. Moreover it is exact when $A^{\mathcal{D}}$ is of finite type. Affinizations of real simple modules and their R -matrices provide a duality functor which enjoys extra good properties (Theorem 4.4).

Several examples of duality functors $\mathfrak{F}^{\mathcal{D}}$ are given in Sections 5 and 6. In Example 5.2, we construct a duality functor $\mathfrak{F}^{\mathcal{D}}$ from the graded module category of the quiver Hecke algebra of type D_{ℓ} to that of type A_{ℓ} . The other examples are in non-symmetric cases. We discuss a duality functor from type C_{ℓ} to type A_{ℓ} in Example 6.2, and ones from types $B_{\ell-1}$ and $A_{\ell-1}$ to type B_{ℓ} in Examples 6.3 and 6.4.

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1. PRELIMINARIES

1.1. **Quantum groups.** Let I be an index set.

Definition 1.1. A Cartan datum is a quadruple $(\mathbf{A}, \mathbf{P}, \Pi, \Pi^{\vee}, (\cdot, \cdot))$ consisting of

- (a) a free abelian group \mathbf{P} , called the weight lattice,
- (b) $\Pi = \{\alpha_i \mid i \in I\} \subset \mathbf{P}$, called the set of simple roots,
- (c) $\Pi^{\vee} = \{h_i \mid i \in I\} \subset \mathbf{P}^{\vee} := \text{Hom}(\mathbf{P}, \mathbb{Z})$, called the set of simple coroots,
- (d) (\cdot, \cdot) is a \mathbb{Q} -valued symmetric bilinear form on \mathbf{P} , which satisfies
 - (1) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ for any $i \in I$,
 - (2) $\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for any $i \in I$ and $\lambda \in \mathbf{P}$,
 - (3) $\mathbf{A} := (\langle h_i, \alpha_j \rangle)_{i,j \in I}$ is a generalized Cartan matrix, i.e., $\langle h_i, \alpha_i \rangle = 2$ for any $i \in I$ and $\langle h_i, \alpha_j \rangle \in \mathbb{Z}_{\leq 0}$ if $i \neq j$,
 - (4) Π is a linearly independent set,
 - (5) for each $i \in I$, there exists $\Lambda_i \in \mathbf{P}$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for any $j \in I$.

Let us write $\mathbf{Q} = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ and $\mathbf{Q}_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$. For $\beta = \sum_{i \in I} k_i \alpha_i \in \mathbf{Q}_+$, set $\text{ht}(\beta) = \sum_{i \in I} k_i$. The Weyl group \mathbf{W} associated with the Cartan datum is the subgroup of $\text{Aut}(\mathbf{P})$ generated by the reflections $\{r_i\}_{i \in I}$ defined by

$$r_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i \quad \text{for } \lambda \in \mathbf{P}.$$

Let \mathfrak{g} be the Kac-Moody algebra associated with a Cartan datum $(\mathbf{A}, \mathbf{P}, \Pi, \Pi^{\vee}, (\cdot, \cdot))$ and Φ_+ the set of positive roots of \mathfrak{g} . We denote by $U_q(\mathfrak{g})$ the corresponding quantum

group, which is an associative algebra over $\mathbb{Q}(q)$ generated by e_i, f_i ($i \in I$) and q^h ($h \in P^\vee$) with certain defining relations (see [3, Chap. 3] for details). Set $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$. We denote by $U_{\mathbf{A}}^-(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $f_i^{(n)} := f_i^n / [n]_i!$ for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$, where $q_i = q^{(\alpha_i, \alpha_i)/2}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^n [k]_i.$$

1.2. Quiver Hecke algebras. Let \mathbf{k} be a field. For $i, j \in I$, we take polynomials $\mathcal{Q}_{i,j}(u, v) \in \mathbf{k}[u, v]$ such that

- (i) $\mathcal{Q}_{i,j}(u, v) = \mathcal{Q}_{j,i}(v, u)$, and
- (ii) it is of the form

$$(1.1) \quad \mathcal{Q}_{i,j}(u, v) = \begin{cases} \sum_{2(\alpha_i, \alpha_j) + p(\alpha_i, \alpha_i) + q(\alpha_j, \alpha_j) = 0} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where $t_{i,j;-a_{ij},0} \in \mathbf{k}^\times$. We set

$$(1.2) \quad \overline{\mathcal{Q}}_{i,j}(u, v, w) = \frac{\mathcal{Q}_{i,j}(u, v) - \mathcal{Q}_{i,j}(w, v)}{u - w} \in \mathbf{k}[u, v, w].$$

For $\beta \in \mathbf{Q}_+$ with $\text{ht}(\beta) = n$, set

$$I^\beta := \left\{ \nu = (\nu_1, \dots, \nu_n) \in I^n \mid \sum_{k=1}^n \alpha_{\nu_k} = \beta \right\}.$$

The symmetric group $\mathfrak{S}_n = \langle s_k \mid k = 1, \dots, n-1 \rangle$ acts on I^β by place permutations.

Definition 1.2. For $\beta \in \mathbf{Q}_+$, the quiver Hecke algebra $R(\beta)$ associated with \mathbf{A} and $(\mathcal{Q}_{i,j}(u, v))_{i,j \in I}$ is the \mathbf{k} -algebra generated by

$$\{e(\nu) \mid \nu \in I^\beta\}, \{x_k \mid 1 \leq k \leq n\}, \{\tau_l \mid 1 \leq l \leq n-1\}$$

satisfying the following defining relations:

$$\begin{aligned}
(1.3) \quad & e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu \in I^\beta} e(\nu) = 1, \quad x_k e(\nu) = e(\nu)x_k, \quad x_k x_l = x_l x_k, \\
& \tau_l e(\nu) = e(s_l(\nu))\tau_l, \quad \tau_k \tau_l = \tau_l \tau_k \text{ if } |k-l| > 1, \\
& \tau_k^2 e(\nu) = \mathcal{Q}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1})e(\nu), \\
& (\tau_k x_l - x_{s_k(l)} \tau_k) e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k \text{ and } \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k+1 \text{ and } \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \\
& (\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) \\
& \quad = \begin{cases} \overline{\mathcal{Q}}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}, x_{k+2})e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The algebra $R(\beta)$ has the \mathbb{Z} -graded algebra structure given by

$$(1.4) \quad \deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}).$$

For $\beta \in \mathbf{Q}_+$, let us denote by $\text{Mod}(R(\beta))$ the category of $R(\beta)$ -modules and by $R(\beta)\text{-mod}$ the category of finite-dimensional $R(\beta)$ -modules.

We denote by $\text{Mod}_{\text{gr}}(R(\beta))$ the category of graded $R(\beta)$ -modules and by $R(\beta)\text{-gmod}$ the category of finite-dimensional graded $R(\beta)$ -modules. We denote by $\text{Mod}_{\text{fg}}(R(\beta))$ the full subcategory of $\text{Mod}_{\text{gr}}(R(\beta))$ consisting of finitely generated graded $R(\beta)$ -modules. Their morphisms are homogeneous of degree zero. Hence, $\text{Mod}(R(\beta))$, $R(\beta)\text{-mod}$, $\text{Mod}_{\text{gr}}(R(\beta))$, $R(\beta)\text{-gmod}$ and $\text{Mod}_{\text{fg}}(R(\beta))$ are abelian categories. We set $\text{Mod}_{\text{gr}}(R) := \bigoplus_{\beta \in \mathbf{Q}^+} \text{Mod}_{\text{gr}}(R(\beta))$, $R\text{-mod} := \bigoplus_{\beta \in \mathbf{Q}^+} R(\beta)\text{-mod}$, etc. The objects of $\text{Mod}_{\text{gr}}(R)$ are sometimes simply called R -modules.

We denote by $R(\beta)\text{-proj}$ the full subcategory of $\text{Mod}_{\text{gr}}(R(\beta))$ consisting of finitely generated projective graded $R(\beta)$ -modules.

Let us denote by q the *grading shift functor*, i.e., $(qM)_k = M_{k-1}$ for a graded module $M = \bigoplus_{k \in \mathbb{Z}} M_k$.

For $\nu \in I^\beta$ and $\nu' \in I^{\beta'}$, let $e(\nu, \nu')$ be the idempotent corresponding to the concatenation $\nu * \nu'$ of ν and ν' , and set

$$e(\beta, \beta') := \sum_{\nu \in I^\beta, \nu' \in I^{\beta'}} e(\nu, \nu').$$

For an $R(\beta)$ -module M and an $R(\beta')$ -module N , we define an $R(\beta + \beta')$ -module $M \circ N$ by

$$M \circ N := R(\beta + \beta')e(\beta, \beta') \otimes_{R(\beta) \otimes R(\beta')} (M \otimes N).$$

We denote by $M \diamond N$ the head of $M \circ N$.

For a graded $R(\beta)$ -module M , the q -character of M is defined by

$$\text{ch}_q(M) := \sum_{\nu \in I^\beta} \dim_q(e(\nu)M)\nu,$$

Here, $\dim_q V := \sum_{k \in \mathbb{Z}} \dim(V_k)q^k$ for a graded vector space $V = \bigoplus_{k \in \mathbb{Z}} V_k$. It is well defined whenever $\dim V_k < \infty$ for all $k \in \mathbb{Z}$.

For $i \in I$, let $L(\alpha_i)$ be the simple graded $R(\alpha_i)$ -module such that $\text{ch}_q L(\alpha_i) = (i)$. For simplicity, we write $L(i)$ for $L(\alpha_i)$ if there is no confusion.

For graded $R(\beta)$ -modules M and N , let $\text{Hom}_{R(\beta)}(M, N)$ be the space of morphisms in $\text{Mod}_{\text{gr}}(R(\beta))$, i.e. the \mathbf{k} -vector space of homogeneous homomorphisms of degree 0, and set

$$(1.5) \quad \text{HOM}_{R(\beta)}(M, N) = \bigoplus_{k \in \mathbb{Z}} \text{HOM}_{R(\beta)}(M, N)_k \quad \text{with } \text{HOM}_{R(\beta)}(M, N)_k := \text{Hom}_{R(\beta)}(q^k M, N).$$

We write $\text{END}_{R(\beta)}(M)$ for $\text{HOM}_{R(\beta)}(M, M)$. When $f \in \text{Hom}_{R(\beta)}(q^k M, N)$, we denote

$$\deg(f) := k.$$

For simplicity, we write $\text{HOM}_R(M, N)$ for $\text{HOM}_{R(\beta)}(M, N)$ if there is no confusion.

We write $[R\text{-proj}]$ and $[R\text{-gmod}]$ for the (split) Grothendieck group of $R\text{-proj}$ and the Grothendieck group of $R\text{-gmod}$. Then, the \mathbb{Z} -grading gives a $\mathbb{Z}[q, q^{-1}]$ -module structure on $[R\text{-proj}]$ and $[R\text{-gmod}]$, and the convolution gives an algebra structure.

Theorem 1.3 ([13, 14, 18]). *There exist algebra isomorphisms*

$$[R\text{-proj}] \simeq U_{\mathbf{A}}^-(\mathfrak{g}), \quad [R\text{-gmod}] \simeq A_q(\mathfrak{g}^+).$$

Here, $A_q(\mathfrak{g}^+) := \{a \in U_q^-(\mathfrak{g}) \mid (a, U_{\mathbf{A}}^-(\mathfrak{g})) \subset \mathbf{A}\}$, where (\cdot, \cdot) is the non-degenerate symmetric bilinear form on $U_q^-(\mathfrak{g})$ defined in [12]. Note that $A_q(\mathfrak{g}^+)$ is an \mathbf{A} -subalgebra of $U_{\mathbf{A}}^-(\mathfrak{g})$ (cf. [9] where $A_q(\mathfrak{g}^+)$ is denoted by $A_q(\mathfrak{n})_{\mathbb{Z}[q^{\pm 1}]}$).

Definition 1.4. *Let \mathbf{c} be a \mathbb{Z} -valued skew-symmetric bilinear form on \mathbf{Q} . If we redefine $\deg(\tau_l e(\nu))$ to be $-(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}) - \mathbf{c}(\alpha_{\nu_l}, \alpha_{\nu_{l+1}})$, then it gives a well-defined \mathbb{Z} -graded algebra structure on $R(\beta)$. We denote by $R_{\mathbf{c}}(\beta)$ the \mathbb{Z} -graded algebra thus defined.*

The usual grading (1.4) is a special case of such a \mathbb{Z} -grading.

We define $R_{\mathbf{c}}(\beta)\text{-gmod}$, $R_{\mathbf{c}}\text{-gmod}$, etc., similarly.

Let us denote by $\text{Mod}_{\text{gr}}(R_{\mathbf{c}}(\beta))[q^{1/2}]$ the category of $(\frac{1}{2}\mathbb{Z})$ -graded modules over $R_{\mathbf{c}}(\beta)$. For $\nu \in I^\beta$ we set

$$H(\nu) = \frac{1}{2} \sum_{1 \leq a < b \leq \text{ht}(\beta)} \mathbf{c}(\alpha_{\nu_a}, \alpha_{\nu_b}).$$

Lemma 1.5. *For $\beta \in \mathbf{Q}_+$ and $M \in \text{Mod}_{\text{gr}}(R(\beta))[q^{1/2}]$, set*

$$(K_{\mathbf{c}}(M))_n = \bigoplus_{\nu \in I^\beta} e(\nu) M_{n-H(\nu)}.$$

Then we have

- (i) $K_{\mathbf{c}}$ is an equivalence of categories from $\text{Mod}_{\text{gr}}(R(\beta))[q^{1/2}]$ to $\text{Mod}_{\text{gr}}(R_{\mathbf{c}}(\beta))[q^{1/2}]$.
- (ii) For $M \in \text{Mod}_{\text{gr}}(R(\beta))[q^{1/2}]$ and $N \in \text{Mod}_{\text{gr}}(R(\gamma))[q^{1/2}]$, we have

$$K_{\mathbf{c}}(M \circ N) \simeq q^{\frac{1}{2}\mathbf{c}(\beta, \gamma)} K_{\mathbf{c}}(M) \circ K_{\mathbf{c}}(N).$$

Proof. (i) We have

$$\begin{aligned} \tau_k e(\nu) (K_{\mathbf{c}}(M))_n &= \tau_k e(\nu) M_{n-H(\nu)} \\ &\subset e(s_k \nu) M_{n-H(\nu) - (\alpha_{\nu_k}, \alpha_{\nu_{k+1}})} \\ &= e(s_k \nu) (K_{\mathbf{c}}(M))_{n-H(\nu) - (\alpha_{\nu_k}, \alpha_{\nu_{k+1}}) + H(s_k \nu)}. \end{aligned}$$

Then (i) follows from

$$H(s_k \nu) - H(\nu) = \frac{1}{2} (\mathbf{c}(\alpha_{\nu_{k+1}}, \alpha_{\nu_k}) - \mathbf{c}(\alpha_{\nu_k}, \alpha_{\nu_{k+1}})) = -\mathbf{c}(\alpha_{\nu_k}, \alpha_{\nu_{k+1}}).$$

(ii) For $\nu \in I^\beta$ and $\mu \in I^\gamma$, we have

$$\begin{aligned} e(\nu) K_{\mathbf{c}}(M)_a \otimes e(\mu) K_{\mathbf{c}}(N)_b &= e(\nu) M_{a-H(\nu)} \otimes e(\mu) N_{b-H(\mu)} \\ &\subset e(\nu * \mu) (M \circ N)_{a+b-H(\nu)-H(\mu)} \\ &= e(\nu * \mu) K_{\mathbf{c}}(M \circ N)_{a+b-H(\nu)-H(\mu)+H(\nu * \mu)}. \end{aligned}$$

Since

$$H(\nu * \mu) - H(\nu) - H(\mu) = \frac{1}{2} \mathbf{c}(\beta, \gamma),$$

we have

$$K_{\mathbf{c}}(M)_a \otimes K_{\mathbf{c}}(N)_b \subset K_{\mathbf{c}}(M \circ N)_{a+b+\frac{1}{2}\mathbf{c}(\beta, \gamma)}.$$

Therefore we have

$$K_{\mathbf{c}}(M)_a \otimes K_{\mathbf{c}}(N)_b \longrightarrow (q^{-\frac{1}{2}\mathbf{c}(\beta, \gamma)} K_{\mathbf{c}}(M \circ N))_{a+b},$$

which induces an isomorphism

$$K_{\mathbf{c}}(M) \circ K_{\mathbf{c}}(N) \xrightarrow{\sim} q^{-\frac{1}{2}\mathbf{c}(\beta, \gamma)} K_{\mathbf{c}}(M \circ N).$$

□

We define the algebra $U_{\mathbf{A}}^-(\mathfrak{g})_{\mathbf{c}}$ as $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} U_{\mathbf{A}}^-(\mathfrak{g})$ with the new multiplication $\circ_{\mathbf{c}}$ given by

$$a \circ_{\mathbf{c}} b = q^{-\frac{1}{2}\mathbf{c}(\alpha, \beta)} ab \quad \text{for } a \in \mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} U_{\mathbf{A}}^-(\mathfrak{g})_{\alpha} \text{ and } b \in \mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} U_{\mathbf{A}}^-(\mathfrak{g})_{\beta}.$$

We define $A_q(\mathfrak{g}^+)_{\mathbf{c}}$ similarly.

Corollary 1.6. *There is a $\mathbb{Z}[q^{\pm 1/2}]$ -algebra isomorphism*

$$\xi_{\mathbf{c}}: A_q(\mathfrak{g}^+)_{\mathbf{c}} \xrightarrow{\sim} \mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} [R_{\mathbf{c}}\text{-gmod}].$$

1.3. Remark on parity. Under hypothesis (d)(1) in Definition 1.1, the category $\text{Mod}_{\text{gr}}(R(\beta))$ is divided into two parts according to the parity of degrees for any $\beta \in \mathbf{Q}_+$.

Lemma 1.7. *Let $\beta \in \mathbf{Q}_+$. Then there exists a map*

$$S: I^{\beta} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

such that

$$S(s_k \nu) = S(\nu) + (\alpha_{\nu_k}, \alpha_{\nu_{k+1}})$$

for any $\nu \in I^{\beta}$ and any integer k with $1 \leq k < \text{ht}(\beta)$.

Proof. Set $n = \text{ht}(\beta)$. Let us choose a total order \prec on I . Then we set

$$S(\nu) := \sum_{1 \leq a < b \leq n, \nu_a \prec \nu_b} (\alpha_{\nu_a}, \alpha_{\nu_b}).$$

Then we have

$$\begin{aligned} S(s_k \nu) &= S(\nu) + (\delta(\nu_{k+1} \prec \nu_k) - \delta(\nu_k \prec \nu_{k+1}))(\alpha_{\nu_k}, \alpha_{\nu_{k+1}}) \\ &\equiv S(\nu) + (1 - \delta(\nu_k = \nu_{k+1}))(\alpha_{\nu_k}, \alpha_{\nu_{k+1}}) \equiv S(\nu) + (\alpha_{\nu_k}, \alpha_{\nu_{k+1}}) \pmod{2}. \end{aligned}$$

Here, for a statement P , we set $\delta(P)$ to be 1 if P is true and 0 if P is false. □

Proposition 1.8. *Let $\beta \in \mathbf{Q}_+$ and $S: I^{\beta} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ be as in Lemma 1.7. Let $\text{Mod}_{\text{gr}}(R(\beta))^S$ be the full subcategory of $\text{Mod}_{\text{gr}}(R(\beta))$ consisting of graded $R(\beta)$ -modules M such that $e(\nu)M_k = 0$ for any $\nu \in I^{\beta}$ and $k \equiv S(\nu) + 1 \pmod{2}$. Then we have*

$$\text{Mod}_{\text{gr}}(R(\beta)) \simeq \text{Mod}_{\text{gr}}(R(\beta))^S \oplus q \text{Mod}_{\text{gr}}(R(\beta))^S.$$

Proof. For any graded $R(\beta)$ -module M and $\varepsilon = 0, 1$ set

$$M^{\varepsilon} := \bigoplus_{\substack{\nu \in I^{\beta}, k \in \mathbb{Z}, \\ k \equiv S(\nu) + \varepsilon \pmod{2}}} e(\nu)M_k.$$

Then, we can see easily that they are $R(\beta)$ -submodules of M and $M = M^0 \oplus M^1$. Moreover, we have $M^\varepsilon \in q^\varepsilon \text{Mod}_{\text{gr}}(R(\beta))^S$. \square

Note that $q^2 \text{Mod}_{\text{gr}}(R(\beta))^S = \text{Mod}_{\text{gr}}(R(\beta))^S$ and

$$(1.6) \quad \text{Hom}_{R(\beta)}(M, N)_k = 0 \text{ if } k \text{ is odd and } M, N \in \text{Mod}_{\text{gr}}(R(\beta))^S.$$

1.4. R -matrices. Let $\beta \in \mathbb{Q}_+$ and $m = \text{ht}(\beta)$. For $k = 1, \dots, m-1$ and $\nu \in I^\beta$, the intertwiner $\varphi_k \in R(\beta)$ is defined by

$$\varphi_k e(\nu) := \begin{cases} (\tau_k x_k - x_k \tau_k) e(\nu) & \text{if } \nu_k = \nu_{k+1}, \\ \tau_k e(\nu) & \text{otherwise.} \end{cases}$$

Lemma 1.9 ([5, Lem. 1.3.1]).

- (i) $\varphi_k^2 e(\nu) = (\mathcal{Q}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) + \delta_{\nu_k, \nu_{k+1}}) e(\nu)$.
- (ii) $\{\varphi_k\}_{1 \leq k \leq m-1}$ satisfies the braid relation.
- (iii) For a reduced expression $w = s_{i_1} \cdots s_{i_t} \in \mathfrak{S}_m$, let $\varphi_w = \varphi_{i_1} \cdots \varphi_{i_t}$. Then φ_w does not depend on the choice of reduced expressions of w .
- (iv) For $w \in \mathfrak{S}_m$ and $1 \leq k \leq m$, we have $\varphi_w x_k = x_{w(k)} \varphi_w$.
- (v) For $w \in \mathfrak{S}_m$ and $1 \leq k < m$, if $w(k+1) = w(k) + 1$, then $\varphi_w \tau_k = \tau_{w(k)} \varphi_w$.
- (vi) $\varphi_{w^{-1}} \varphi_w e(\nu) = \prod_{a < b, w(a) > w(b)} (\mathcal{Q}_{\nu_a, \nu_b}(x_a, x_b) + \delta_{\nu_a, \nu_b}) e(\nu)$.

For $m, n \in \mathbb{Z}_{\geq 0}$, let $w[m, n]$ be the element of \mathfrak{S}_{m+n} defined by

$$w[m, n](k) = \begin{cases} k + n & \text{if } 1 \leq k \leq m, \\ k - m & \text{if } m < k \leq m + n. \end{cases}$$

Let M be an $R(\beta)$ -module with $\text{ht}(\beta) = m$ and N an $R(\beta')$ -module with $\text{ht}(\beta') = n$. The $R(\beta) \otimes R(\beta')$ -linear map $M \otimes N \rightarrow N \circ M$ given by $u \otimes v \mapsto \varphi_{w[m, n]}(v \otimes u)$ can be extended to the $R(\beta + \beta')$ -module homomorphism

$$(1.7) \quad R_{M, N} : M \circ N \rightarrow N \circ M.$$

For $\beta = \sum_{k=1}^m \alpha_{i_k}$, we set $\text{supp}(\beta) := \{i_k \mid 1 \leq k \leq m\}$.

Definition 1.10. The quiver Hecke algebra $R(\beta)$ is said to be symmetric if $\mathcal{Q}_{i, j}(u, v)$ is a polynomial in $u - v$ for all $i, j \in \text{supp}(\beta)$.

Suppose that $R(\beta)$ is symmetric. Let z be an indeterminate. For an $R(\beta)$ -module M , we define an $R(\beta)$ -module structure on $M_z := \mathbf{k}[z] \otimes_{\mathbf{k}} M$ by

$$(1.8) \quad \begin{aligned} e(\nu)(a \otimes u) &= a \otimes e(\nu)u, & x_j(a \otimes u) &= (za) \otimes u + a \otimes x_j u, \\ \tau_k(a \otimes u) &= a \otimes (\tau_k u), \end{aligned}$$

for $\nu \in I^n$, $a \in \mathbf{k}[z]$ and $u \in M$. We call M_z the *affinization* of M . For a non-zero $R(\beta)$ -module M and a non-zero $R(\beta')$ -module N , let s be the order of zeros of $R_{M_z, N_{z'}} : M_z \circ N_{z'} \rightarrow N_{z'} \circ M_z$, and

$$R_{M_z, N_{z'}}^{\text{norm}} := (z' - z)^{-s} R_{M_z, N_{z'}}.$$

We define $\mathbf{r}_{M, N} : M \circ N \longrightarrow N \circ M$ by

$$\mathbf{r}_{M, N} := R_{M_z, N_{z'}}^{\text{norm}}|_{z=z'=0}.$$

We put $R(\beta)[z_1, \dots, z_k] := \mathbf{k}[z_1, \dots, z_k] \otimes_{\mathbf{k}} R(\beta)$. For simplicity, we write $R[z_1, \dots, z_k]$ for $R(\beta)[z_1, \dots, z_k]$ if there is no afraid of confusion.

Theorem 1.11 ([5, Section 1]). *Suppose that $R(\beta)$ and $R(\beta')$ are symmetric. Let M be a non-zero $R(\beta)$ -module and N a non-zero $R(\beta')$ -module.*

- (i) $R_{M_z, N_{z'}}^{\text{norm}}$ and $\mathbf{r}_{M, N}$ are non-zero.
- (ii) $R_{M_z, N_{z'}}^{\text{norm}}$ and $\mathbf{r}_{M, N}$ satisfy the braid relations.
- (iii) Set

$$A = \sum_{\mu \in I^\beta, \nu \in I^{\beta'}} \left(\prod_{1 \leq a \leq m, 1 \leq b \leq n, \mu_a \neq \nu_b} \mathcal{Q}_{\mu_a, \nu_b}(x_a \boxtimes e(\beta'), e(\beta) \boxtimes x_b) \right) e(\mu) \boxtimes e(\nu).$$

Then, A is in the center of $R(\beta) \otimes R(\beta')$, and

$$R_{N_{z'}, M_z} R_{M_z, N_{z'}}(u \otimes v) = A(u \otimes v)$$

for $u \in M_z$ and $v \in N_{z'}$.

- (iv) If M and N are simple modules, then

$$\begin{aligned} \text{END}_{R(\beta+\beta')[z, z']}(M_z \circ N_{z'}) &\simeq \mathbf{k}[z, z'], \\ \text{HOM}_{R(\beta+\beta')[z, z']}(M_z \circ N_{z'}, N_{z'} \circ M_z) &\simeq \mathbf{k}[z, z'] R_{M_z, N_{z'}}^{\text{norm}}. \end{aligned}$$

2. AFFINIZATION

2.1. Definition of affinization.

Definition 2.1. For any $i \in I$ and $\beta \in \mathbf{Q}_+$ with $\text{ht}(\beta) = m$, we set

$$\mathfrak{p}_{i, \beta} = \sum_{\nu \in I^\beta} \left(\prod_{a \in [1, m], \nu_a = i} x_a \right) e(\nu),$$

where $[1, m] = \{1, 2, \dots, m\}$.

Note that $\mathfrak{p}_{i, \beta}$ belongs to the center of $R(\beta)$. If there is no afraid of confusion, we simply write \mathfrak{p}_i for $\mathfrak{p}_{i, \beta}$.

Definition 2.2. Let $\beta \in \mathbb{Q}_+$ and \bar{M} a simple $R(\beta)$ -module. An affinization $\mathbf{M} := (\mathbf{M}, z_{\mathbf{M}})$ of \bar{M} is an $R(\beta)$ -module \mathbf{M} with an injective homogeneous endomorphism $z_{\mathbf{M}}$ of \mathbf{M} of degree $d_{\mathbf{M}} \in \mathbb{Z}_{>0}$ and an isomorphism $\mathbf{M}/z_{\mathbf{M}}\mathbf{M} \xrightarrow{\sim} \bar{M}$ satisfying the following conditions:

- (a) \mathbf{M} is a finitely generated free module over the polynomial ring $\mathbf{k}[z_{\mathbf{M}}]$,
- (b) $\mathfrak{p}_i \mathbf{M} \neq 0$ for any $i \in I$.

If it satisfies moreover the following additional condition, we say that \mathbf{M} is a strong affinization:

- (c) the exact sequence $0 \rightarrow z_{\mathbf{M}}\mathbf{M}/z_{\mathbf{M}}^2\mathbf{M} \rightarrow \mathbf{M}/z_{\mathbf{M}}^2\mathbf{M} \rightarrow \mathbf{M}/z_{\mathbf{M}}\mathbf{M} \rightarrow 0$ of $R(\beta)$ -modules does not split.

We say that an affinization is even if $d_{\mathbf{M}}$ is even.

Let us denote by $\pi_{\mathbf{M}}: \mathbf{M} \twoheadrightarrow \bar{M}$ the composition $\mathbf{M} \twoheadrightarrow \mathbf{M}/z_{\mathbf{M}}\mathbf{M} \xrightarrow{\sim} \bar{M}$.

Remark 2.3.

- (i) Condition (a) is equivalent to the condition
 - (a') The degree of \mathbf{M} is bounded from below, that is, $\mathbf{M}_n = 0$ for $n \ll 0$.
 Moreover, under these equivalent conditions, we have

$$\mathrm{ch}_q(\mathbf{M}) = (1 - q^{d_{\mathbf{M}}})^{-1} \mathrm{ch}_q(\bar{M}).$$

Note that a finitely generated R -module M satisfies the condition (a').

- (ii) The non-splitness condition (c) is equivalent to saying that $z_{\mathbf{M}}\mathbf{M}/z_{\mathbf{M}}^2\mathbf{M}$ is a unique proper $R(\beta)$ -submodule of $\mathbf{M}/z_{\mathbf{M}}^2\mathbf{M}$.
- (iii) If $R(\beta)$ is a symmetric quiver Hecke algebra, then \bar{M}_z is a strong affinization of any simple $R(\beta)$ -module \bar{M} for $\beta \neq 0$.

Example 2.4. (i) For $i \in I$, $\mathbf{M} := L(i)_z \circ L(i)$ is not an affinization of $\bar{M} := L(i) \circ L(i)$

In fact, conditions (a) and (c) in Definition 2.2 hold but condition (b) does not.

- (ii) Let $(\mathbf{M}, z_{\mathbf{M}})$ be an affinization of \bar{M} . Assume that $d_{\mathbf{M}} = ab$ for $a, b \in \mathbb{Z}_{>0}$ and let z be an indeterminate of homogeneous degree b . Let $\mathbf{k}[z_{\mathbf{M}}] \rightarrow \mathbf{k}[z]$ be the algebra homomorphism given by $z_{\mathbf{M}} \mapsto z^a$. Then $(\mathbf{k}[z] \otimes_{\mathbf{k}[z_{\mathbf{M}}]} \mathbf{M}, z)$ is an affinization of \bar{M} . If $a > 1$ then it is never a strong affinization, because

$$(\mathbf{k}[z] \otimes_{\mathbf{k}[z_{\mathbf{M}}]} \mathbf{M}) / (z^a \mathbf{k}[z] \otimes_{\mathbf{k}[z_{\mathbf{M}}]} \mathbf{M}) \simeq (\mathbf{k}[z] / \mathbf{k}[z]z^a) \otimes_{\mathbf{k}} \bar{M}$$

is a semisimple $R(\beta)$ -module.

As seen in the proposition below, every affinization is essentially even.

Proposition 2.5. *Let $(\mathbf{M}, z_{\mathbf{M}})$ be an affinization of a simple module \bar{M} . Assume that the homogeneous degree $d_{\mathbf{M}}$ of $z_{\mathbf{M}}$ is odd. Then there exists an $R(\beta)$ -submodule \mathbf{M}' of \mathbf{M} such that*

- (i) $z_{\mathbf{M}}^2 \mathbf{M}' \subset \mathbf{M}'$, and $(\mathbf{M}', z_{\mathbf{M}}^2)$ is an affinization of \bar{M} ,
- (ii) $\mathbf{M} \simeq \mathbf{k}[z_{\mathbf{M}}] \otimes_{\mathbf{k}[z_{\mathbf{M}}^2]} \mathbf{M}'$ as an $R(\beta)[z_{\mathbf{M}}]$ -module.

Proof. Let $\text{Mod}_{\text{gr}}(R(\beta)) \simeq \text{Mod}_{\text{gr}}(R(\beta))^S \oplus q \text{Mod}_{\text{gr}}(R(\beta))^S$ be the decomposition in Proposition 1.8. We may assume that \bar{M} belongs to $\text{Mod}_{\text{gr}}(R(\beta))^S$. Let $\mathbf{M} = \mathbf{M}' \oplus \mathbf{M}''$ be the decomposition with $\mathbf{M}' \in \text{Mod}_{\text{gr}}(R(\beta))^S$ and $\mathbf{M}'' \in q \text{Mod}_{\text{gr}}(R(\beta))^S$. Then $z_{\mathbf{M}} \mathbf{M}' \subset \mathbf{M}''$ and $z_{\mathbf{M}} \mathbf{M}'' \subset \mathbf{M}'$ by (1.6). Hence, we have the decomposition

$$\mathbf{M}/z_{\mathbf{M}} \mathbf{M} = (\mathbf{M}'/z_{\mathbf{M}} \mathbf{M}'') \oplus (\mathbf{M}''/z_{\mathbf{M}} \mathbf{M}'),$$

which implies that $\mathbf{M}'/z_{\mathbf{M}} \mathbf{M}'' \simeq \bar{M}$ and $\mathbf{M}'' = z_{\mathbf{M}} \mathbf{M}'$. Thus we obtain the desired result. \square

2.2. Strong affinization. Note that Lemmas 2.6 and 2.7 below hold without assumption (b) in Definition 2.2.

Lemma 2.6. *Assume that*

$$(2.1)_{\text{strong}} \quad \begin{cases} \beta \in \mathbf{Q}_+ \text{ and } (\mathbf{M}, z_{\mathbf{M}}) \text{ is a strong affinization of a simple } R(\beta)\text{-module } \bar{M}, \\ z_{\mathbf{M}} \text{ has homogeneous degree } d_{\mathbf{M}} \in \mathbb{Z}_{>0}, \text{ and } \pi_{\mathbf{M}}: \mathbf{M} \rightarrow \bar{M} \text{ is a canonical projection.} \end{cases}$$

- (i) *The head of the R -module \mathbf{M} is isomorphic to \bar{M} , or equivalently $z_{\mathbf{M}} \mathbf{M}$ is a unique maximal $R(\beta)$ -module.*
- (ii) *Let $s := \min \{m \in \mathbb{Z} \mid \mathbf{M}_m \neq 0\}$, and u a non-zero element of \mathbf{M}_s . Then, $\mathbf{M} = R(\beta)u$.*
- (iii) $\text{END}_{R(\beta)}(\mathbf{M}) \simeq \mathbf{k}[z_{\mathbf{M}}] \text{id}_{\mathbf{M}}$.

Proof. (i) Let S be a simple module and $\varphi: \mathbf{M} \rightarrow S$ be an epimorphism. By consideration of the homogeneous degree, we may assume that $\varphi(z_{\mathbf{M}}^k \mathbf{M}) = 0$ for $k \gg 0$. Let us take $k \geq 0$ such that $\varphi(z_{\mathbf{M}}^k \mathbf{M}) = S$ and $\varphi(z_{\mathbf{M}}^{k+1} \mathbf{M}) = 0$. Since $z_{\mathbf{M}}^k \mathbf{M}/z_{\mathbf{M}}^{k+1} \mathbf{M} \simeq \mathbf{M}/z_{\mathbf{M}} \mathbf{M}$ is simple, φ induces an isomorphism $z_{\mathbf{M}}^k \mathbf{M}/z_{\mathbf{M}}^{k+1} \mathbf{M} \xrightarrow{\sim} S$. It is enough to show that $k = 0$.

If $k > 0$ then we have a commutative diagram

$$(2.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & z_{\mathbf{M}}^k \mathbf{M}/z_{\mathbf{M}}^{k+1} \mathbf{M} & \longrightarrow & z_{\mathbf{M}}^{k-1} \mathbf{M}/z_{\mathbf{M}}^{k+1} \mathbf{M} & \longrightarrow & z_{\mathbf{M}}^{k-1} \mathbf{M}/z_{\mathbf{M}}^k \mathbf{M} \longrightarrow 0 \\ & & & \searrow \sim & \downarrow \varphi & & \\ & & & & S & & \end{array}$$

Hence the first row of the above diagram is a split exact sequence, which contradicts Definition 2.2 (c).

(ii) Since $u \notin z_M M$, (i) implies that $M = R(\beta)u$.

(iii) Let $f \in \text{END}_{R(\beta)}(M)$ be a homogeneous endomorphism of degree ℓ . Assume that $f(M) \subset z_M^k M$ for $k \in \mathbb{Z}_{\geq 0}$. We shall show $f \in \mathbf{k}[z_M] \text{id}_M$ by the descending induction on k . If $d_M k > \ell$, then f should be 0 since $f(u) \notin z_M^k M$ if $f(u) \neq 0$. Here u is as in (ii).

Suppose that $d_M k \leq \ell$. As \bar{M} is the head of M , the composition $M \xrightarrow{z_M^{-k} f} M \xrightarrow{\pi_M} \bar{M}$ decomposes as $M \xrightarrow{\pi_M} \bar{M} \rightarrow \bar{M}$. Hence the composition must be equal to $c\pi_M$ for some $c \in \mathbf{k}$, which yields

$$(z_M^{-k} f - c \text{id}_M)(M) \subset z_M M.$$

Therefore, we have $(f - cz_M^k)(M) \subset z_M^{k+1} M$, and the induction proceeds. \square

2.3. Normalized R-matrices.

Lemma 2.7. *Assume that*

$$(2.3)_{\text{weak}} \quad \begin{cases} \beta \in \mathbf{Q}^+ \text{ and } (M, z_M) \text{ is an affinization of a simple } R(\beta)\text{-module } \bar{M}, z_M \\ \text{has homogeneous degree } d_M \in \mathbb{Z}_{>0}, \text{ and } \pi_M: M \rightarrow \bar{M} \text{ is a canonical} \\ \text{projection.} \end{cases}$$

(i) $\text{END}_{R(\beta)[z_M]}(M) \simeq \mathbf{k}[z_M] \text{id}_M$.

(ii) For any $i \in I$, there exist $c_i \in \mathbf{k}^\times$ and $d_i \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{p}_i|_M = c_i z_M^{d_i}$.

Proof. (i) The proof is similar to that of Lemma 2.6 (iii). Let $f \in \text{END}_{R(\beta)[z_M]}(M)$ be a homogeneous endomorphism of degree ℓ . Suppose that $f(M) \subset z_M^k M$ for $k \in \mathbb{Z}_{\geq 0}$. We shall show $f \in \mathbf{k}[z_M] \text{id}_M$ by the descending induction on k .

We have $f = 0$ if $d_M k > \ell$ by the degree consideration. If $d_M k \leq \ell$, then the endomorphism $z_M^{-k} f$ induces an endomorphism of \bar{M} . Hence it must be equal to $c \text{id}_{\bar{M}}$ for some $c \in \mathbf{k}$. Then $(f - cz_M^k)(M) \subset z_M^{k+1} M$, and the induction proceeds.

(ii) The assertion follows from (i) immediately. \square

Lemma 2.8. *Let β , M and \bar{M} be as in (2.3)_{weak}. Assume further that $\beta \neq 0$. Then M is a finitely generated $R(\beta)$ -module.*

Proof. Since $\beta \neq 0$, there exists $i \in I$ such that $\mathfrak{p}_{i,\beta}$ has a positive degree. Then there exists $m > 0$ such that $z_M^m \in \mathbf{k}(\mathfrak{p}_{i,\beta}|_M) \subset \text{END}_R(M)$. Since M is finitely generated over $\mathbf{k}[z_M^m]$, we obtain the desired result. \square

Lemma 2.9. *Let β , M and \bar{M} be as in (2.3)_{weak}. Let $\gamma \in \mathbf{Q}_+$ and $N \in R(\gamma)\text{-gmod}$.*

(i) *The homomorphisms*

$$R_{M[z_M^{-1}], N}: M[z_M^{-1}] \circ N \rightarrow N \circ M[z_M^{-1}] \text{ and } R_{N, M[z_M^{-1}]}: N \circ M[z_M^{-1}] \rightarrow M[z_M^{-1}] \circ N$$

are isomorphisms. Here, $M[z_M^{-1}] = \mathbf{k}[z_M, z_M^{-1}] \otimes_{\mathbf{k}[z_M]} M$.

- (ii) If N is a simple module, there exist $c \in \mathbf{k}^\times$ and $d \in \mathbb{Z}_{\geq 0}$ such that $R_{N,M} \circ R_{M,N} = c(z_M^d \circ N)$ and $R_{M,N} \circ R_{N,M} = c(N \circ z_M^d)$.

Proof. (i) is an immediate consequence of (ii). Let us show (ii). Set $m = \text{ht}(\beta)$ and $n = \text{ht}(\gamma)$. Then, $(R_{N,M} \circ R_{M,N})|_{M \otimes N}$ is given by

$$\sum_{\nu \in I^{\beta+\gamma}} \left(\prod_{1 \leq a \leq m < b \leq n, \nu_a \neq \nu_b} \mathcal{Q}_{\nu_a, \nu_b}(x_a, x_b) \right) e(\nu).$$

Since any element in the center of $R(\gamma)$ with positive degree acts by zero on N , it is equal to

$$\sum_{\nu \in I^{\beta+\gamma}} \left(\prod_{1 \leq a \leq m < b \leq n, \nu_a \neq \nu_b} \mathcal{Q}_{\nu_a, \nu_b}(x_a, 0) \right) e(\nu).$$

Hence it is equal to a product of $\mathfrak{p}_{i,\beta}|_M$'s up to a constant multiple. Hence Lemma 2.7 (ii) implies the desired result. \square

Let M and \bar{M} be as in (2.3)_{weak}, and let $N \in R\text{-gmod}$ be a non-zero module. Let s be the largest integer such that $R_{M,N}(M \circ N) \subset z_M^s N \circ M$. Then we set

$$R_{M,N}^{\text{norm}} = z_M^{-s} R_{M,N}: M \circ N \rightarrow N \circ M.$$

We denote by

$$\mathbf{r}_{\bar{M},N}: \bar{M} \circ N \rightarrow N \circ \bar{M}$$

the homomorphism induced by $R_{M,N}^{\text{norm}}$. By the definition, $\mathbf{r}_{\bar{M},N}$ never vanishes. We set $R_{M,N}^{\text{norm}} = 0$, and $\mathbf{r}_{\bar{M},N} = 0$ when $N = 0$.

We define $R_{N,M}^{\text{norm}}$ and $\mathbf{r}_{N,\bar{M}}$ similarly.

The arguments in [8, 9, 11] still work well under these assumptions, and we obtain similar results. We list some of them without repeating the proof. A simple module S is called *real* if $S \circ S$ is simple.

Proposition 2.10 ([11, Th. 3.2, Prop. 3.8], [9, Prop. 4.5], [8, Cor. 3.2]). *Assume that*

$$(2.4) \quad \begin{cases} \text{(a) } M \text{ and } N \text{ are simple } R\text{-modules,} \\ \text{(b) one of them is real simple and also admits an affinization.} \end{cases}$$

- (i) $M \circ N$ has a simple head and a simple socle. Moreover, $\text{Im}(\mathbf{r}_{M,N})$ is equal to the head of $M \circ N$ and the socle of $N \circ M$.
- (ii) We have

$$\text{Hom}_R(M \circ N, M \circ N) = \mathbf{k} \text{id}_{M \circ N}$$

and

$$\mathrm{Hom}_R(M \circ N, N \circ M) = \mathbf{k} \mathbf{r}_{M,N}.$$

(iii) $M \diamond N$ appears only once in a Jordan-Hölder series of $M \circ N$ in R -mod.

Proposition 2.11. *Let \mathbf{M} and \bar{M} be as in (2.3)_{weak}, and let N be a simple R -module. Assume that \bar{M} is real. Then we have*

(i)

$$(2.5) \quad \mathrm{Hom}_{R[z_M]}(\mathbf{M} \circ N, \mathbf{M} \circ N) = \mathbf{k}[z_M] \mathrm{id}_{\mathbf{M} \circ N}$$

and

$$(2.6) \quad \mathrm{Hom}_{R[z_M]}(N \circ \mathbf{M}, N \circ \mathbf{M}) = \mathbf{k}[z_M] \mathrm{id}_{N \circ \mathbf{M}}.$$

(ii) $\mathrm{Hom}_{R[z_M]}(\mathbf{M} \circ N, N \circ \mathbf{M})$ and $\mathrm{Hom}_{R[z_M]}(N \circ \mathbf{M}, \mathbf{M} \circ N)$ are free $\mathbf{k}[z_M]$ -modules of rank one.

Proof. (i) Let us first show (2.5). The idea for the proof is similar to that of Lemma 2.6 (iii).

Let $f \in \mathrm{Hom}_{R[z_M]}(\mathbf{M} \circ N, \mathbf{M} \circ N)$ of homogeneous degree ℓ . We know that $f(\mathbf{M} \circ N) \subset z_M^s \mathbf{M} \circ N$ for some $s \in \mathbb{Z}_{\geq 0}$. We shall show $f \in \mathbf{k}[z_M] \mathrm{id}_{\mathbf{M} \circ N}$ by the descending induction on s . If $s \gg 0$, then f should be zero by the degree consideration. Now, we consider $z_M^{-s} f$. As $z_M^{-s} f$ induces an endomorphism of $\bar{M} \circ N$, by Proposition 2.10 (ii), it is equal to $c \mathrm{id}_{\bar{M} \circ N}$ for $c \in \mathbf{k}$. Hence we have

$$(f - cz_M^s)(\mathbf{M} \circ N) \subset z_M^{s+1} \mathbf{M} \circ N.$$

Thus, the induction hypothesis implies that $f - cz_M^s \in \mathbf{k}[z_M] \mathrm{id}_{\bar{M} \circ N}$. The proof of (2.6) is similar.

(ii) By Lemma 2.9, we have an $R[z_M]$ -linear monomorphism $N \circ \mathbf{M} \hookrightarrow \mathbf{M} \circ N$. Hence we have

$$\mathrm{Hom}_{R[z_M]}(\mathbf{M} \circ N, N \circ \mathbf{M}) \hookrightarrow \mathrm{Hom}_{R[z_M]}(\mathbf{M} \circ N, \mathbf{M} \circ N) \simeq \mathbf{k}[z_M].$$

As $\mathrm{Hom}_{R[z_M]}(\mathbf{M} \circ N, N \circ \mathbf{M})$ is non-zero, $\mathrm{Hom}_{R[z_M]}(\mathbf{M} \circ N, N \circ \mathbf{M})$ is a free $\mathbf{k}[z_M]$ -module of rank one. \square

Proposition 2.12. *We assume that*

$$(2.7) \quad \begin{cases} \text{(a) } (\mathbf{M}, z_M) \text{ and } (\mathbf{N}, z_N) \text{ are affinizations of simple modules } \bar{M} \text{ and } \bar{N}, \\ \text{respectively,} \\ \text{(b) one of } \bar{M} \text{ and } \bar{N} \text{ is real.} \end{cases}$$

Then we have

- (i) $\text{HOM}_{R[z_M, z_N]}(\mathbf{M} \circ \mathbf{N}, \mathbf{M} \circ \mathbf{N}) = \mathbf{k}[z_M, z_N] \text{id}_{\mathbf{M} \circ \mathbf{N}}$,
- (ii) $\text{HOM}_{R[z_M, z_N]}(\mathbf{M} \circ \mathbf{N}, \mathbf{N} \circ \mathbf{M})$ is a free $\mathbf{k}[z_M, z_N]$ -module of rank one.

Proof. (i) Assume that \bar{M} is real simple. The other case can be proved similarly.

Let f be a homogeneous element of $\text{HOM}_{R[z_M, z_N]}(\mathbf{M} \circ \mathbf{N}, \mathbf{M} \circ \mathbf{N})$ of degree ℓ . Assuming that $\text{Im}(f) \subset z_N^k(\mathbf{M} \circ \mathbf{N})$, we shall show $f \in \mathbf{k}[z_M, z_N] \text{id}_{\mathbf{M} \circ \mathbf{N}}$ by the descending induction on k . If $k \gg 0$, then f should be zero by the homogeneous-degree consideration. We now consider $z_N^{-k}f$. The $R[z_M, z_N]$ -linear homomorphism $z_N^{-k}f: \mathbf{M} \circ \mathbf{N} \rightarrow \mathbf{M} \circ \mathbf{N}$ induces an $R[z_M]$ -linear homomorphism $\mathbf{M} \circ \bar{N} \rightarrow \mathbf{M} \circ \bar{N}$. By Proposition 2.11, it is equal to $\varphi(z_M) \text{id}_{\mathbf{M} \circ \bar{N}}$ for some $\varphi(z_M) \in \mathbf{k}[z_M]$. Hence we have

$$\text{Im}(f - z_N^k \varphi(z_M) \text{id}_{\mathbf{M} \circ \mathbf{N}}) \subset z_N^{k+1} \mathbf{M} \circ \mathbf{N},$$

which implies

$$f - z_N^k \varphi(z_M) \text{id}_{\mathbf{M} \circ \mathbf{N}} \in \mathbf{k}[z_M, z_N] \text{id}_{\mathbf{M} \circ \mathbf{N}}$$

by the induction hypothesis.

- (ii) An injective morphism $R_{M,N}: \mathbf{N} \circ \mathbf{M} \hookrightarrow \mathbf{M} \circ \mathbf{N}$ induces an injective homomorphism

$$\text{HOM}_{R[z_M, z_N]}(\mathbf{M} \circ \mathbf{N}, \mathbf{N} \circ \mathbf{M}) \hookrightarrow \text{HOM}_{R[z_M, z_N]}(\mathbf{M} \circ \mathbf{N}, \mathbf{M} \circ \mathbf{N}) \simeq \mathbf{k}[z_M, z_N].$$

Since the non-zero $\mathbf{k}[z_M, z_N]$ -module $L := \text{HOM}_{R[z_M, z_N]}(\mathbf{M} \circ \mathbf{N}, \mathbf{N} \circ \mathbf{M})$ satisfies the condition:

if non-zero elements $a, b \in \mathbf{k}[z_M, z_N]$ are prime to each other, then we have $aL \cap bL = abL$,

we conclude that L is a free $\mathbf{k}[z_M, z_N]$ -module of rank one. \square

We define $R_{M,N}^{\text{norm}}$ as a generator of the $\mathbf{k}[z_M, z_N]$ -module $\text{HOM}_{R[z_M, z_N]}(\mathbf{M} \circ \mathbf{N}, \mathbf{N} \circ \mathbf{M})$. It is uniquely determined up to a constant multiple. We call it a *normalized R-matrix*.

Theorem 2.13. *Assume (2.7). Then, $R_{M,N}^{\text{norm}}|_{z_M=z_N=0}: \bar{M} \circ \bar{N} \rightarrow \bar{N} \circ \bar{M}$ does not vanish and is equal to $\mathbf{r}_{\bar{M}, \bar{N}}$ up to a constant multiple.*

Proof. Since any simple R -module is absolutely simple, we may assume that the base field \mathbf{k} is algebraically closed without loss of generality.

By Proposition 2.10 (ii), we have

$$(2.8) \quad \text{HOM}_R(\bar{M} \circ \bar{N}, \bar{N} \circ \bar{M}) = \mathbf{k} \mathbf{r}_{\bar{M}, \bar{N}}$$

for a non-zero $\mathbf{r}_{\bar{M}, \bar{N}} \in \text{HOM}_R(\bar{M} \circ \bar{N}, \bar{N} \circ \bar{M})$. Let ℓ be the homogeneous degree of $\mathbf{r}_{\bar{M}, \bar{N}}$.

For $a \in \mathbb{Z}$, let $\mathbf{k}[z_M, z_N]_a$ be the homogeneous part of $\mathbf{k}[z_M, z_N]$ of degree a and set $\mathbf{k}[z_M, z_N]_{\geq a} = \bigoplus_{k \geq a} \mathbf{k}[z_M, z_N]_k$. Let $c \in \mathbb{Z}$ be the largest integer such that

$$R_{M,N}^{\text{norm}}(\mathbf{M} \circ \mathbf{N}) \subset \mathbf{k}[z_M, z_N]_{\geq c}(\mathbf{N} \circ \mathbf{M}).$$

Then $R_{\mathbf{M}, \mathbf{N}}^{\text{norm}}$ induces a non-zero map

$$\varphi: \bar{M} \circ \bar{N} \longrightarrow \left(\mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]_{\geq c}(\mathbf{N} \circ \mathbf{M}) \right) / \left(\mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]_{\geq c+1}(\mathbf{N} \circ \mathbf{M}) \right).$$

Since $\mathbf{N} \circ \mathbf{M}$ is a free $\mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]$ -module, we have

$$\left(\mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]_{\geq c}(\mathbf{N} \circ \mathbf{M}) \right) / \left(\mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]_{\geq c+1}(\mathbf{N} \circ \mathbf{M}) \right) \simeq \mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]_c \otimes (\bar{N} \circ \bar{M}).$$

By (2.8), there exists a non-zero $f(z_{\mathbf{M}}, z_{\mathbf{N}}) \in \mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]_c$ such that

$$\varphi(u) = f(z_{\mathbf{M}}, z_{\mathbf{N}}) \mathbf{r}_{\bar{M}, \bar{N}}(u) \text{ for any } u \in \bar{M} \circ \bar{N}.$$

Hence, the homogeneous degree of $R_{\mathbf{M}, \mathbf{N}}^{\text{norm}}$ is equal to $c + \ell$, and

$$R_{\mathbf{M}, \mathbf{N}}^{\text{norm}}(\mathbf{M} \circ \mathbf{N}) \subset f(z_{\mathbf{M}}, z_{\mathbf{N}})(\mathbf{N} \circ \mathbf{M}) + \mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]_{\geq c+1}(\mathbf{N} \circ \mathbf{M}).$$

Let us show that $f(z_{\mathbf{M}}, z_{\mathbf{N}})$ is a constant function (i.e., $c = 0$). Assuming that $c > 0$, let us take a prime divisor $a(z_{\mathbf{M}}, z_{\mathbf{N}})$ of $f(z_{\mathbf{M}}, z_{\mathbf{N}})$. Let (x, y) be an arbitrary point of \mathbf{k}^2 such that $a(x, y) = 0$. Let $d_{\mathbf{M}}$ and $d_{\mathbf{N}}$ be the homogeneous degree of $z_{\mathbf{M}}$ and $z_{\mathbf{N}}$, respectively, and let z be an indeterminate of homogeneous degree one. Then $a(xz^{d_{\mathbf{M}}}, yz^{d_{\mathbf{N}}}) = f(xz^{d_{\mathbf{M}}}, yz^{d_{\mathbf{N}}}) = 0$. Let $\mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}] \rightarrow \mathbf{k}[z]$ be the map obtained by the substitution $z_{\mathbf{M}} = xz^{d_{\mathbf{M}}}$ and $z_{\mathbf{N}} = yz^{d_{\mathbf{N}}}$. Set

$$K = \mathbf{k}[z]_{\mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]} \otimes (\mathbf{M} \circ \mathbf{N}), \quad K' = \mathbf{k}[z]_{\mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]} \otimes (\mathbf{N} \circ \mathbf{M}), \quad R' = \mathbf{k}[z]_{\mathbf{k}[z_{\mathbf{M}}, z_{\mathbf{N}}]} \otimes R_{\mathbf{M}, \mathbf{N}}^{\text{norm}}.$$

Then we obtain the map

$$R': K \rightarrow z^{c+1} K'.$$

Note that $K/zK \simeq \bar{M} \circ \bar{N}$ and $K'/zK' \simeq \bar{N} \circ \bar{M}$. We shall show $R'(K) \subset z^k K'$ for any $k \geq c + 1$ by induction on k . Assume that $k \geq c + 1$ and $R'(K) \subset z^k K'$. Then the morphism $\bar{M} \circ \bar{N} \rightarrow \bar{N} \circ \bar{M}$ induced by $z^{-k} R'$ is equal to $b \mathbf{r}_{\bar{M}, \bar{N}}$ for some $b \in \mathbf{k}$. If $b \neq 0$, then the homogeneous degree of R' is equal to $k + \ell > c + \ell$, which is a contradiction. Thus $b = 0$ and $R'(K) \subset z^{k+1} K'$. Hence the induction proceeds, and we conclude that

$$R_{\mathbf{M}, \mathbf{N}}^{\text{norm}}|_{z_{\mathbf{M}}=xz^{d_{\mathbf{M}}}, z_{\mathbf{N}}=yz^{d_{\mathbf{N}}}} = 0$$

for any $(x, y) \in \mathbf{k}^2$ such that $a(x, y) = 0$, which implies $R_{\mathbf{M}, \mathbf{N}}^{\text{norm}}$ is divided by $a(z_{\mathbf{M}}, z_{\mathbf{N}})$. This is a contradiction.

Therefore f is a constant function, and $R_{\mathbf{M}, \mathbf{N}}^{\text{norm}}$ induces $\mathbf{r}_{\bar{M}, \bar{N}}$ (up to a constant multiple) after the specialization $z_{\mathbf{M}} = z_{\mathbf{N}} = 0$. \square

Corollary 2.14. *Assume (2.7). If \bar{M} is real, then $R_{\mathbf{M}, \mathbf{N}}^{\text{norm}}|_{z_{\mathbf{N}}=0} = R_{\mathbf{M}, \bar{N}}^{\text{norm}}$ (up to a constant multiple).*

Lemma 2.15. *Assume (2.7). Then there exists a homogeneous element $f(z_M, z_N)$ such that*

- (i) $R_{N,M}^{\text{norm}} \circ R_{M,N}^{\text{norm}} = f(z_M, z_N) \text{id}_{M \circ N}$ and $R_{M,N}^{\text{norm}} \circ R_{N,M}^{\text{norm}} = f(z_M, z_N) \text{id}_{N \circ M}$,
- (ii) $f(z_M, 0)$ and $f(0, z_N)$ are non-zero.

Proof. It follows from Proposition 2.12, Corollary 2.14 and Lemma 2.9. \square

Lemma 2.16. *Let (M, z_M) and (N, z_N) be affinizations of simple modules \bar{M} and \bar{N} , respectively. Assume that either \bar{M} or \bar{N} is real, and $\bar{M} \circ \bar{N} \simeq \bar{N} \circ \bar{M}$. Let d be a common divisor of the homogeneous degree d_M of z_M and d_N , that of z_N . Let z be an indeterminate of homogeneous degree d and let $\mathbf{k}[z_M, z_N] \rightarrow \mathbf{k}[z]$ be the algebra homomorphism given by $z_M \mapsto z^{d_M/d}$ and $z_N \mapsto z^{d_N/d}$. Then, $\mathbf{k}[z] \otimes_{\mathbf{k}[z_M, z_N]} (M \circ N)$ is an affinization of $\bar{M} \circ \bar{N}$.*

Proof. By the condition, $\bar{M} \circ \bar{N}$ is simple. Condition (a) in Definition 2.2 is obvious. Condition (b) follows from $\mathfrak{p}_i|_{M \circ N} = (\mathfrak{p}_i|_M) \circ (\mathfrak{p}_i|_N)$. \square

Proposition 2.17. *Let M and \bar{M} be as in (2.3)_{weak}. Assume that \bar{M} is real. We normalize $R_{M,M}^{\text{norm}}$ so that it induces $\text{id}_{\bar{M} \circ \bar{M}}$ after the specialization $z_M \circ \text{id}_M = \text{id}_M \circ z_M = 0$. Then, we have*

- (i) $(R_{M,M}^{\text{norm}} - \text{id}_{M \circ M})(M \circ M) \subset (z_M \circ \text{id}_M - \text{id}_M \circ z_M)(M \circ M)$.
- (ii) $R_{M,M}^{\text{norm}} \circ R_{M,M}^{\text{norm}} = \text{id}_{M,M}$.

Proof. (i) To avoid possible confusions, let (N, z_N) be a copy of (M, z_M) and we regard $R_{M,M}^{\text{norm}}$ as a homomorphism $M \circ N \rightarrow N \circ M$. We denote by $\iota: M \circ N \xrightarrow{\sim} N \circ M$ the identity. We regard $M \circ N$ and $N \circ M$ as $R[z_M, z_N]$ -modules. Then, $R_{M,N}^{\text{norm}}$ commutes with z_M and z_N , but ι does not. Precisely, we have $\iota \circ z_M = z_N \circ \iota$ and $\iota \circ z_N = z_M \circ \iota$.

By the assumption we have

$$(R_{M,N}^{\text{norm}} - \iota)(M \circ N) \subset (z_N N) \circ M + N \circ (z_M M).$$

Let z be another indeterminate with the same homogeneous degree d_M , and let

$$\mathbf{k}[z_M, z_N] \rightarrow \mathbf{k}[z]$$

be the algebra homomorphism given by $z_M \mapsto z$ and $z_N \mapsto z$. Then, Lemma 2.16 implies that $K := \mathbf{k}[z] \otimes_{\mathbf{k}[z_M, z_N]} (M \circ N)$ is an affinization of $\bar{M} \circ \bar{M}$. The homomorphisms $R_{M,N}^{\text{norm}}$ and ι induce $R[z]$ -linear endomorphisms R' and ι' of K . By Lemma 2.7 (i), R' and ι' are powers of z up to a constant multiple. Since they are $\text{id}_{\bar{M} \circ \bar{M}}$ after the specialization $z = 0$, we conclude that $R' = \iota'$, which completes the proof.

- (ii) The assertion follows from Lemma 2.15 immediately. \square

Example 2.18. Let $i \in I$. Let $P(i^n)$ be a projective cover of the simple module $L(i)^{\circ n}$. Then $P(i^n)$ is an $R(n\alpha_i)$ -module generated by an element u of degree 0 with the defining relation $\tau_k u = 0$ ($1 \leq k < n$). Let $e_k(x_1, \dots, x_n)$ be the elementary symmetric function of degree k . The center of $R(n\alpha_i)$ is equal to $\mathbf{k}[e_k(x_1, \dots, x_n) \mid k = 1, \dots, n] = \mathbf{k}[x_1, \dots, x_n]^{\mathfrak{S}_n}$. Then we have

$$L(i)^{\circ n} \simeq P(i^n) / \left(\sum_{k=1}^n R(n\alpha_i) e_k(x_1, \dots, x_n) u \right).$$

Set

$$K(i^n) := P(i^n) / \left(\sum_{k=1}^{n-1} R(n\alpha_i) e_k(x_1, \dots, x_n) u \right),$$

and define $\mathbf{z}_{K(i^n)} \in \text{END}_{R(n\alpha_i)}(K(i^n))$ by

$$\mathbf{z}_{K(i^n)} u = e_n(x_1, \dots, x_n) u.$$

Then $(K(i^n), \mathbf{z}_{K(i^n)})$ is a strong affinization of $L(i)^{\circ n}$. Note that $\mathfrak{p}_i|_{K(i^n)} = \mathbf{z}_{K(i^n)}$. The homogeneous degree of $\mathbf{z}_{K(i^n)}$ is $n(\alpha_i, \alpha_i)$.

3. ROOT MODULES

In this section, we shall review results of McNamara ([16]) and Brundan-Kleshchev-McNamara ([2]). Throughout this section, we assume that the Cartan matrix \mathbf{A} is of finite type. Fix a reduced expression $w_0 = r_{i_1} r_{i_2} \dots r_{i_N}$ of the longest element $w_0 \in \mathbf{W}$. This expression gives a convex total order \prec on the set Φ_+ of positive roots: $\alpha_{i_1} \prec r_{i_1} \alpha_{i_2} \prec \dots \prec r_{i_1} \dots r_{i_{N-1}} \alpha_{i_N}$. To each positive root $\beta \in \Phi_+$, McNamara defined a simple $R(\beta)$ -module $L(\beta)$, which he called the *cuspidal module* ([15, 16]).

Lemma 3.1 ([16, Lem. 3.4]). *For any $\beta \in \mathbf{Q}_+$, $L(\beta)$ is a real simple module.*

Lemma 3.2 ([2, Lem. 3.2]). *For $n \geq 0$, there exist unique (up to isomorphism) $R(\beta)$ -modules $\Delta_n(\beta)$ with $\Delta_0(\beta) = 0$ such that there are short exact sequences*

$$\begin{aligned} 0 \longrightarrow q_\beta^{2(n-1)} L(\beta) &\xrightarrow{i_n} \Delta_n(\beta) \xrightarrow{p_n} \Delta_{n-1}(\beta) \longrightarrow 0, \\ 0 \longrightarrow q_\beta^2 \Delta_{n-1}(\beta) &\xrightarrow{j_n} \Delta_n(\beta) \xrightarrow{q_n} L(\beta) \longrightarrow 0 \quad \text{for } n \geq 1, \end{aligned}$$

where $q_\beta = q^{(\beta, \beta)/2}$. Moreover,

- (i) $[\Delta_n(\beta)] = \frac{1-q_\beta^{2n}}{1-q_\beta^2} [L(\beta)]$,
- (ii) $\Delta_n(\beta)$ is a cyclic module with simple head isomorphic to $L(\beta)$ and socle isomorphic to $q_\beta^{2(n-1)} L(\beta)$,

(iii) for $n \geq 1$, we have

$$\mathrm{Ext}_{R(\beta)}^k(\Delta_n(\beta), L(\beta)) \simeq \begin{cases} q_\beta^{-2n} \mathbf{k} & \text{if } k = 1, \\ 0 & \text{if } k \geq 2. \end{cases}$$

Define the root module

$$\Delta(\beta) := \varprojlim_n \Delta_n(\beta).$$

Theorem 3.3 ([2, Th. 3.3]). *There is a short exact sequence*

$$0 \longrightarrow q_\beta^2 \Delta(\beta) \xrightarrow{z_\beta} \Delta(\beta) \longrightarrow L(\beta) \longrightarrow 0.$$

Moreover,

- (i) $\Delta(\beta)$ is a cyclic module with $[\Delta(\beta)] = [L(\beta)]/(1 - q_\beta^2)$,
- (ii) $L(\beta)$ is the head of $\Delta(\beta)$,
- (iii) $\mathrm{END}_{R(\beta)}(\Delta(\beta)) \simeq \mathbf{k}[z_\beta]$.

Corollary 3.4 ([2, Cor. 3.5]). *Any finitely generated graded $R(\beta)$ -module with all simple subquotients isomorphic to $L(\beta)$ (up to a grading shift) is a finite direct sum of grade-shifted copies of the indecomposable modules $\Delta_n(\beta)$ ($n \geq 1$) and $\Delta(\beta)$.*

Proposition 3.5. *For any $\beta \in \Phi_+$, $(\Delta(\beta), z_\beta)$ is a strong affinization of $L(\beta)$.*

Proof. We can easily check that conditions (a) and (c) in Definition 2.2 are satisfied.

We shall show (b) by induction on $\mathrm{ht}(\beta)$. If β is a simple root, then (b) is obvious. Assume that $\mathrm{ht}(\beta) > 1$. Then, by [2, Lemma 4.9, Theorem 4.10], there exist $\alpha, \gamma \in \Phi_+$ such that $\alpha + \gamma = \beta$ and there exists an exact sequence

$$0 \rightarrow q^{-(\alpha, \gamma)} \Delta(\gamma) \circ \Delta(\alpha) \xrightarrow{\varphi} \Delta(\alpha) \circ \Delta(\gamma) \rightarrow [1 + p] \Delta(\beta) \rightarrow 0.$$

Here p is some non-negative integer and $[1 + p]$ is the q -integer with respect to the short root. Moreover φ is given by

$$(3.1) \quad \varphi(u \otimes v) = \tau_{w[m, n]}(v \otimes u)$$

for any $u \in \Delta(\gamma)$ and $v \in \Delta(\alpha)$. Here $m = \mathrm{ht}(\alpha)$ and $n = \mathrm{ht}(\gamma)$.

By the induction hypothesis, $(\Delta(\alpha), z_\alpha)$ and $(\Delta(\gamma), z_\gamma)$ are affinizations. By (3.1), we see that φ commutes with z_α and z_γ . Then $\varphi = a(z_\alpha, z_\gamma) R_{\Delta(\gamma), \Delta(\alpha)}^{\mathrm{norm}}$ for some $a(z_\alpha, z_\gamma) \in \mathbf{k}[z_\alpha, z_\gamma]$ by Proposition 2.12.

Note that $\mathfrak{p}_i|_{\Delta(\alpha) \circ \Delta(\gamma)} = (\mathfrak{p}_i|_{\Delta(\alpha)}) \circ (\mathfrak{p}_i|_{\Delta(\gamma)})$, and $\mathfrak{p}_i|_{\Delta(\alpha)} = c_1 z_\alpha^{s_1}$ and $\mathfrak{p}_i|_{\Delta(\gamma)} = c_2 z_\gamma^{s_2}$ for $c_1, c_2 \in \mathbf{k}^\times$ and $s_1, s_2 \in \mathbb{Z}_{\geq 0}$. Hence, if (b) failed, then $(z_\alpha z_\gamma)^s|_{\Delta(\beta)} = 0$ for some $s > 0$. Then we have

$$(z_\alpha z_\gamma)^s \Delta(\alpha) \circ \Delta(\gamma) \subset \mathrm{Im}(\varphi) \subset \mathrm{Im}(R_{\Delta(\gamma), \Delta(\alpha)}^{\mathrm{norm}}).$$

Let us take $f(z_\alpha, z_\gamma) \in \mathbf{k}[z_\alpha, z_\gamma]$ such that $R_{\Delta(\gamma), \Delta(\alpha)}^{\text{norm}} R_{\Delta(\alpha), \Delta(\gamma)}^{\text{norm}} = f(z_\alpha, z_\gamma) \text{id}_{\Delta(\alpha) \circ \Delta(\gamma)}$. Then we have

$$(z_\alpha z_\beta)^s \text{Im}(R_{\Delta(\alpha), \Delta(\gamma)}^{\text{norm}}) \subset f(z_\alpha, z_\gamma) \Delta(\gamma) \circ \Delta(\alpha).$$

By Lemma 2.15, we have $f(z_\alpha, 0) \neq 0$ and $f(0, z_\gamma) \neq 0$, which implies

$$\text{Im}(R_{\Delta(\alpha), \Delta(\gamma)}^{\text{norm}}) \subset f(z_\alpha, z_\gamma) \Delta(\gamma) \circ \Delta(\alpha).$$

Therefore $f(z_\alpha, z_\gamma)^{-1} R_{\Delta(\alpha), \Delta(\gamma)}^{\text{norm}}$ is well defined, which implies that f is an invertible element of \mathbf{k} . Hence $R_{\Delta(\alpha), \Delta(\gamma)}^{\text{norm}}$ is an isomorphism. Then $L(\alpha) \circ L(\gamma)$ is simple, which is a contradiction. \square

Note that $[L(\beta)] \in [R\text{-gmod}] \simeq A_q(\mathfrak{g}^+)$ coincides with the dual PBW vector $E^*(\beta) \in A_q(\mathfrak{g}^+)$. It is known that $\{E^*(m_1, \dots, m_N)\}_{(m_1, \dots, m_N) \in \mathbb{Z}_{\geq 0}^N}$ is a basis of $A_q(\mathfrak{g}^+)$, which is called the *dual PBW basis*. Here, we set

$$E^*(m_1, \dots, m_N) := \left(q_{\beta_1}^{m_1(m_1-1)/2} E^*(\beta_1)^{m_1} \right) \cdots \left(q_{\beta_N}^{m_N(m_N-1)/2} E^*(\beta_N)^{m_N} \right)$$

with $\beta_k := r_{i_1} \cdots r_{i_{k-1}} \alpha_{i_k}$ and $q_\beta = q^{(\beta, \beta)/2}$ ($k = 1, \dots, N$). On the other hand, $E(\beta) = \frac{E^*(\beta)}{(E^*(\beta), E^*(\beta))}$ is called the PBW vector and

$$\{E(\beta_1)^{(m_1)} \cdots E(\beta_N)^{(m_N)}\}_{(m_1, \dots, m_N) \in \mathbb{Z}_{\geq 0}^N}$$

is the basis of $U_{\mathbf{A}}^-(\mathfrak{g})$ called the *PBW basis*. Here $E(\beta)^{(m)} = \frac{E(\beta)^m}{[m]_i!}$ with $i \in I$ such that $(\beta, \beta) = (\alpha_i, \alpha_i)$. Note that the PBW basis and the dual PBW basis are dual to each other.

4. THE DUALITY FUNCTOR

4.1. Duality data. Let R be the quiver Hecke algebra associated with a generalized Cartan matrix \mathbf{A} and polynomials $\mathcal{Q}_{i,j}(u, v)$.

Definition 4.1. Let J be a finite index set. We say that $\mathcal{D} = \{\beta_j, M_j, \mathbf{z}_j, \mathbf{r}_j, \mathbf{R}_{j,k}\}_{j,k \in J}$ is a duality datum if $\beta_j \in \mathbf{Q}_+ \setminus \{0\}$, $M_j \in \text{Mod}_{\text{gr}}(R(\beta_j))$ and homogeneous homomorphisms

$$(4.1) \quad \begin{aligned} \mathbf{z}_j &\in \text{END}_{R(\beta_j)}(M_j), \quad \mathbf{r}_j \in \text{END}_{R(2\beta_j)}(M_j \circ M_j), \\ \mathbf{R}_{j,k} &\in \text{HOM}_{R(\beta_j + \beta_k)}(M_j \circ M_k, M_k \circ M_j) \quad \text{for } j, k \in J \end{aligned}$$

satisfy the following conditions:

(F-1) For $j \in J$, $\deg \mathbf{z}_j \in 2\mathbb{Z}_{>0}$. In addition, M_j is a finitely generated free module over the polynomial ring $\mathbf{k}[\mathbf{z}_j]$.

(\mathcal{F} -2) For $j \in J$, we have $r_j \in \text{END}_{R(2\beta_j)}(M_j \circ M_j)_{-\deg z_j}$ and

$$R_{j,j} = (z_j \circ M_j - M_j \circ z_j) r_j + \text{id}_{M_j \circ M_j}.$$

(\mathcal{F} -3) For $k, l \in J$,

(a) $(z_l \circ M_k) R_{k,l} = R_{k,l} (M_k \circ z_l)$ in $\text{HOM}_{R(\beta_k + \beta_l)}(M_k \circ M_l, M_l \circ M_k)$,

(b) $(M_l \circ z_k) R_{k,l} = R_{k,l} (z_k \circ M_l)$ in $\text{HOM}_{R(\beta_k + \beta_l)}(M_k \circ M_l, M_l \circ M_k)$.

(\mathcal{F} -4) There exist polynomials $\mathcal{Q}_{k,l}^{\mathcal{D}}(u, v) \in \mathbf{k}[u, v]$ ($k, l \in J$) such that

(a) $\mathcal{Q}_{k,k}^{\mathcal{D}}(u, v) = 0$, and $\mathcal{Q}_{k,l}^{\mathcal{D}}(u, v)$ ($k \neq l$) is of the form

$$\sum_{\deg R_{k,l} + \deg R_{l,k} - p \deg z_k - q \deg z_l = 0} t_{k,l;p,q} u^p v^q,$$

where $t_{k,l}; (\deg R_{k,l} + \deg R_{l,k}) / \deg z_k, 0 \in \mathbf{k}^\times$,

(b) $\mathcal{Q}_{k,l}^{\mathcal{D}}(u, v) = \mathcal{Q}_{l,k}^{\mathcal{D}}(v, u)$,

(c) $R_{l,k} R_{k,l} = \begin{cases} 1 & \text{if } k = l, \\ \mathcal{Q}_{k,l}^{\mathcal{D}}(z_k \circ M_l, M_k \circ z_l) & \text{if } k \neq l. \end{cases}$

(\mathcal{F} -5) For any $j, k, l \in J$,

$$(R_{k,l} \circ M_j)(M_k \circ R_{j,l})(R_{j,k} \circ M_l) = (M_l \circ R_{j,k})(R_{j,l} \circ M_k)(M_j \circ R_{k,l})$$

holds in $\text{HOM}_{R(\beta_j + \beta_k + \beta_l)}(M_j \circ M_k \circ M_l, M_l \circ M_k \circ M_j)$.

For simplicity, we write shortly $\{M_j, z_j, R_{j,k}\}_{j,k \in J}$ for $\{\beta_j, M_j, z_j, r_j, R_{j,k}\}_{j,k \in J}$ if there is no afraid of confusion.

We now construct a Cartan datum corresponding to the duality datum \mathcal{D} as follows. Let $\{\alpha_j^{\mathcal{D}}\}_{j \in J}$ be the simple roots. Then we define the weight lattice $\mathbf{P}^{\mathcal{D}}$ by $\mathbf{P}^{\mathcal{D}} = \mathbf{Q}^{\mathcal{D}} := \bigoplus_{j \in J} \mathbb{Z} \alpha_j^{\mathcal{D}}$, and define a symmetric bilinear form on $\mathbf{P}^{\mathcal{D}}$ by

$$(4.2) \quad (\alpha_j^{\mathcal{D}}, \alpha_k^{\mathcal{D}}) = \begin{cases} \deg z_j & \text{if } j = k, \\ -(\deg \mathcal{Q}_{j,k}^{\mathcal{D}}(z_j, z_k)) / 2 = -(\deg R_{j,k} + \deg R_{k,j}) / 2 & \text{otherwise.} \end{cases}$$

Define $h_j^{\mathcal{D}}$ by (d) (2) in Definition 1.1. Then the corresponding generalized Cartan matrix $\mathbf{A}^{\mathcal{D}} := (a_{jk}^{\mathcal{D}})_{j,k \in J}$ is given by $a_{jk}^{\mathcal{D}} = \frac{2(\alpha_j^{\mathcal{D}}, \alpha_k^{\mathcal{D}})}{(\alpha_j^{\mathcal{D}}, \alpha_j^{\mathcal{D}})}$. Since $\mathcal{Q}_{j,k}^{\mathcal{D}}(z_j, 0) \in \mathbf{k}^\times z_j^{-a_{jk}^{\mathcal{D}}}$ for $j \neq k$, $-a_{jk}^{\mathcal{D}}$ is a non-negative integer. Therefore, $\mathbf{A}^{\mathcal{D}}$ is a generalized Cartan matrix. We now define $R^{\mathcal{D}}$ as the quiver Hecke algebra corresponding to the datum $\{\mathcal{Q}_{j,k}^{\mathcal{D}}\}_{j,k \in J}$.

We now have two different quiver Hecke algebras R and $R^{\mathcal{D}}$. To distinguish them, we write

$$x_k^{\mathcal{D}} \ (1 \leq k \leq \text{ht}(\gamma)) \text{ and } \tau_l^{\mathcal{D}} \ (1 \leq l \leq \text{ht}(\gamma) - 1)$$

for the generators x_k ($1 \leq k \leq \text{ht}(\gamma)$) and τ_l ($1 \leq l \leq \text{ht}(\gamma) - 1$) of $R^{\mathcal{D}}(\gamma)$ ($\gamma \in \mathcal{Q}_+^{\mathcal{D}}$).

The \mathbb{Z} -grading on $R^{\mathcal{D}}(\gamma)$ is give as follows:

$$\deg(e(\mu)) = 0, \quad \deg(e(\mu)x_k^{\mathcal{D}}) = \deg z_{\mu_k}, \quad \deg(e(\mu)\tau_l^{\mathcal{D}}) = \begin{cases} -\deg z_{\mu_l} & \text{if } \mu_l = \mu_{l+1}, \\ \deg R_{\mu_l, \mu_{l+1}} & \text{if } \mu_l \neq \mu_{l+1}, \end{cases}$$

which is well-defined (see Definition 1.4).

Let $\gamma \in \mathcal{Q}_+^{\mathcal{D}}$ with $m = \text{ht}(\gamma)$, and define

$$\Delta^{\mathcal{D}}(\gamma) := \bigoplus_{\mu \in J^\gamma} \Delta_\mu^{\mathcal{D}},$$

where

$$\Delta_\mu^{\mathcal{D}} := M_{\mu_1} \circ M_{\mu_2} \circ \cdots \circ M_{\mu_m} \quad \text{for } \mu = (\mu_1, \mu_2, \dots, \mu_m) \in J^\gamma.$$

Let

$$\phi: \mathcal{Q}^{\mathcal{D}} \rightarrow \mathcal{Q}$$

be the linear map defined by $\phi(\alpha_j^{\mathcal{D}}) = \beta_j$ for $j \in J$. Then, it is clear that $\Delta^{\mathcal{D}}(\gamma)$ is a left $R(\phi(\gamma))$ -module.

We define a right $R^{\mathcal{D}}(\gamma)$ -module structure on $\Delta^{\mathcal{D}}(\gamma)$ as follows:

- (a) $e(\mu)$ is the projection to the component $\Delta_\mu^{\mathcal{D}}$,
- (b) the action of $x_k^{\mathcal{D}}$ on $\Delta_\mu^{\mathcal{D}}$ is given by $M_{\mu_1} \circ \cdots \circ M_{\mu_{k-1}} \circ z_{\mu_k} \circ M_{\mu_{k+1}} \circ \cdots \circ M_{\mu_m}$,
- (c) if $\mu_k \neq \mu_{k+1}$, the action of $\tau_k^{\mathcal{D}}$ on $\Delta_\mu^{\mathcal{D}}$ is given by

$$M_{\mu_1} \circ \cdots \circ M_{\mu_{k-1}} \circ R_{\mu_k, \mu_{k+1}} \circ M_{\mu_{k+2}} \circ \cdots \circ M_{\mu_m},$$

- (d) if $\mu_k = \mu_{k+1}$, the action of $\tau_k^{\mathcal{D}}$ on $\Delta_\mu^{\mathcal{D}}$ is given by

$$M_{\mu_1} \circ \cdots \circ M_{\mu_{k-1}} \circ r_{\mu_k} \circ M_{\mu_{k+2}} \circ \cdots \circ M_{\mu_m}.$$

Theorem 4.2. *The right $R^{\mathcal{D}}(\gamma)$ -module structure on $\Delta^{\mathcal{D}}(\gamma)$ is well-defined.*

Proof. Since the proof is easy and similar to the arguments in [5], we omit it. \square

By the construction, the right $R^{\mathcal{D}}(\gamma)$ -module action commutes with the left $R(\phi(\gamma))$ -module action, which means that

$$\Delta^{\mathcal{D}}(\gamma) \text{ has an } (R(\phi(\gamma)), R^{\mathcal{D}}(\gamma))\text{-bimodule structure.}$$

We now define the functor

$$\mathfrak{F}_\gamma^{\mathcal{D}}: \text{Mod}_{\text{gr}}(R^{\mathcal{D}}(\gamma)) \longrightarrow \text{Mod}_{\text{gr}}(R(\phi(\gamma)))$$

by

$$\mathfrak{F}_\gamma^{\mathcal{D}}(M) := \Delta^{\mathcal{D}}(\gamma) \otimes_{R^{\mathcal{D}}(\gamma)} M.$$

Set

$$\mathfrak{F}^{\mathcal{D}} = \bigoplus_{\gamma \in \mathbb{Q}_+^{\mathcal{D}}} \mathfrak{F}_{\gamma}^{\mathcal{D}}.$$

For $j \in J$, we shall write $L^{\mathcal{D}}(j)$ for the simple $R^{\mathcal{D}}(\alpha_j^{\mathcal{D}})$ -module $R^{\mathcal{D}}(\alpha_j^{\mathcal{D}})/R^{\mathcal{D}}(\alpha_j^{\mathcal{D}})x_1^{\mathcal{D}}$.

Theorem 4.3. *Let $\mathcal{D} = \{\beta_j, M_j, z_j, r_j, R_{j,k}\}_{j,k \in J}$ be a duality datum.*

(i) *The functor*

$$\mathfrak{F}^{\mathcal{D}} : \text{Mod}_{\text{gr}}(R^{\mathcal{D}}) \longrightarrow \text{Mod}_{\text{gr}}(R)$$

is a tensor functor.

(ii) *For $j \in J$,*

$$\mathfrak{F}^{\mathcal{D}}(R^{\mathcal{D}}(\alpha_j^{\mathcal{D}})) \simeq M_j \text{ and } \mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j)) \simeq M_j/z_j M_j.$$

(iii) *If $A^{\mathcal{D}}$ is of finite type, then the functor $\mathfrak{F}^{\mathcal{D}}$ is exact.*

(iv) *If a graded $R^{\mathcal{D}}(\gamma)$ -module L is finite-dimensional, then so is $\mathfrak{F}^{\mathcal{D}}(L)$. Thus, we have the induced functor*

$$\mathfrak{F}^{\mathcal{D}} : R^{\mathcal{D}}\text{-gmod} \longrightarrow R\text{-gmod}.$$

Proof. Since the proof is easy and similar to [5], we omit it. □

4.2. Construction of duality data from affinizations. Let J be a finite index set.

Let $\{\beta_j, M_j, z_j\}_{j \in J}$ be a datum such that

(a) $\beta_j \in \mathbb{Q}_+ \setminus \{0\}$,

(b) (M_j, z_j) is an even affinization of a real simple $R(\beta_j)$ -module $\bar{M}_j := M_j/z_j M_j$.

Then we take $R_{j,k}$ as follows:

(c) $R_{j,k} = R_{M_j, M_k}^{\text{norm}}$. Furthermore, we normalize $R_{j,j}$ such that $R_{j,k}|_{z_j=z_k=0} = \text{id}_{\bar{M}_j \circ \bar{M}_j}$ when $j = k$.

Then, Proposition 2.17 implies that

$$(4.3) \quad r_j := (z_j \circ M_j - M_j \circ z_j)^{-1} (R_{j,j} - \text{id}_{M_j \circ M_j})$$

is a well-defined endomorphism of $M_j \circ M_j$.

Note that for any $\{\beta_j, M_j, z_j\}_{j \in J}$ satisfying (a) and (b), we can always choose $R_{j,k}$'s. Moreover, $R_{j,j}$ is unique and $R_{j,k}$ ($j \neq k$) is unique up to constant multiple.

Theorem 4.4. *Under the above assumptions (a), (b), (c), we have the following.*

(i) *The datum $\mathcal{D} = \{\beta_j, M_j, z_j, r_j, R_{j,k}\}_{j,k \in J}$ is a duality datum.*

(ii) *Assume that $A^{\mathcal{D}}$ is of finite type. Then, we have the following.*

- (a) $\mathfrak{F}^{\mathcal{D}}(M)$ is either a simple module or vanishes for any simple $R^{\mathcal{D}}$ -module M . Moreover, if M is a real simple module and $\mathfrak{F}^{\mathcal{D}}(M)$ is non-zero, then $\mathfrak{F}^{\mathcal{D}}(M)$ is real.
- (b) Let $(\mathbf{N}, z_{\mathbf{N}})$ is an affinization of a simple $R^{\mathcal{D}}$ -module \bar{N} . If $\mathfrak{F}^{\mathcal{D}}(\bar{N})$ is simple, then $(\mathfrak{F}^{\mathcal{D}}(\mathbf{N}), \mathfrak{F}^{\mathcal{D}}(z_{\mathbf{N}}))$ is an affinization of $\mathfrak{F}^{\mathcal{D}}(\bar{N})$.
- (c) Let M and N be simple $R^{\mathcal{D}}$ -modules, and assume that one of them is real and also admits an affinization. Then $\mathfrak{F}^{\mathcal{D}}(M \diamond N)$ is zero or isomorphic to $\mathfrak{F}^{\mathcal{D}}(M) \diamond \mathfrak{F}^{\mathcal{D}}(N)$.

Proof. (i) Let us prove that \mathcal{D} is a duality datum. Since axioms $(\mathcal{F}-1)$ – $(\mathcal{F}-4)$ are obvious, we only gives the proof of the braid relation $(\mathcal{F}-5)$:

$$(4.4) \quad R_{jk} \circ R_{ik} \circ R_{ij} = R_{ij} \circ R_{ik} \circ R_{jk}$$

as a morphism from $\mathbf{M}_i \circ \mathbf{M}_j \circ \mathbf{M}_k \rightarrow \mathbf{M}_k \circ \mathbf{M}_j \circ \mathbf{M}_i$ for $i, j, k \in J$. By the definition, we have $R_{\mathbf{M}_i, \mathbf{M}_j} = a(z_i, z_j) R_{i,j}$ for a non-zero polynomial $a(z_i, z_j)$. The R -matrices $R_{\mathbf{M}_i, \mathbf{M}_j}$ satisfy the braid relation:

$$R_{\mathbf{M}_j, \mathbf{M}_k} \circ R_{\mathbf{M}_i, \mathbf{M}_k} \circ R_{\mathbf{M}_i, \mathbf{M}_j} = R_{\mathbf{M}_i, \mathbf{M}_j} \circ R_{\mathbf{M}_i, \mathbf{M}_k} \circ R_{\mathbf{M}_j, \mathbf{M}_k}.$$

The calculation

$$\begin{aligned} R_{\mathbf{M}_j, \mathbf{M}_k} \circ R_{\mathbf{M}_i, \mathbf{M}_k} \circ R_{\mathbf{M}_i, \mathbf{M}_j} &= a(z_j, z_k) R_{j,k} \circ a(z_i, z_k) R_{i,k} \circ a(z_i, z_j) R_{i,j} \\ &= a(z_j, z_k) a(z_i, z_k) a(z_i, z_j) R_{j,k} \circ R_{i,k} \circ R_{i,j}, \end{aligned}$$

and a similar calculation for $R_{\mathbf{M}_i, \mathbf{M}_j} \circ R_{\mathbf{M}_i, \mathbf{M}_k} \circ R_{\mathbf{M}_j, \mathbf{M}_k}$ show that

$$a(z_j, z_k) a(z_i, z_k) a(z_i, z_j) (R_{jk} \circ R_{ik} \circ R_{ij} - R_{ij} \circ R_{ik} \circ R_{jk}) = 0.$$

Hence we obtain (4.4).

(ii) (a) Let us prove that $\mathfrak{F}^{\mathcal{D}}(M)$ is a simple module or zero for a simple $R^{\mathcal{D}}(\gamma)$ -module M by induction on $\text{ht}(\gamma)$. Let us assume $M \simeq N \diamond L^{\mathcal{D}}(j)$ for some $j \in J$ and a simple $R^{\mathcal{D}}(\gamma - \alpha_j^{\mathcal{D}})$ -module N . By the induction hypothesis, $\mathfrak{F}^{\mathcal{D}}(N)$ is a simple module or zero. Let $r: N \circ L^{\mathcal{D}}(j) \rightarrow L^{\mathcal{D}}(j) \circ N$ be a non-zero homomorphism of $R^{\mathcal{D}}(\gamma)$ -modules. Then $\text{Im}(r)$ is isomorphic to $N \diamond L^{\mathcal{D}}(j)$. Since $\mathfrak{F}^{\mathcal{D}}$ is exact, $\mathfrak{F}^{\mathcal{D}}(\text{Im}(r)) \simeq \text{Im}(\mathfrak{F}^{\mathcal{D}}(r)) \simeq \mathfrak{F}^{\mathcal{D}}(M)$. If $\mathfrak{F}^{\mathcal{D}}(N) \simeq 0$, then $\mathfrak{F}^{\mathcal{D}}(M) \simeq 0$. Assume that $\mathfrak{F}^{\mathcal{D}}(N)$ is a simple module. Then $\text{Im}(\mathfrak{F}^{\mathcal{D}}(r))$ is isomorphic to $\mathfrak{F}^{\mathcal{D}}(N) \diamond \mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j))$ or 0 according that $\mathfrak{F}^{\mathcal{D}}(r)$ is non-zero or zero by Proposition 2.10.

If M is real simple and $\mathfrak{F}^{\mathcal{D}} M$ is simple, then $(\mathfrak{F}^{\mathcal{D}} M) \circ (\mathfrak{F}^{\mathcal{D}} M) \simeq \mathfrak{F}^{\mathcal{D}}(M \circ M)$ is simple and hence $\mathfrak{F}^{\mathcal{D}} M$ is real.

Thus we obtain (ii) (a).

(ii) (b) Let \bar{N} be a simple $R^{\mathcal{D}}(\gamma)$ -module and set $m = \text{ht}(\gamma)$. We put $N_{\mathfrak{F}} = \mathfrak{F}^{\mathcal{D}}(N)$ and $z_{\mathfrak{F}} = \mathfrak{F}^{\mathcal{D}}(z_N)$. Applying the functor $\mathfrak{F}^{\mathcal{D}}$ to the exact sequence

$$0 \longrightarrow N \xrightarrow{z_N} N \longrightarrow \bar{N} \longrightarrow 0,$$

we obtain the exact sequence

$$0 \longrightarrow N_{\mathfrak{F}} \xrightarrow{z_{\mathfrak{F}}} N_{\mathfrak{F}} \longrightarrow \mathfrak{F}^{\mathcal{D}}(\bar{N}) \longrightarrow 0.$$

Thus, we have an injective homogeneous endomorphism $z_{\mathfrak{F}}$ of $N_{\mathfrak{F}}$ and $N_{\mathfrak{F}}/z_{\mathfrak{F}}N_{\mathfrak{F}} \simeq \mathfrak{F}^{\mathcal{D}}(\bar{N})$. Since M_j is a finitely generated $R(\beta_j)$ -module for any j by Lemma 2.8, $N_{\mathfrak{F}}$ is a finitely generated graded R -module and $\mathfrak{F}^{\mathcal{D}}(\bar{N})$ is a finite-dimensional R -module. Hence, condition (a) of Definition 2.2 holds (see Remark 2.3 (i)).

Let us show (b) of Definition 2.2. Let $i \in I$. By Lemma 2.7 (ii), for any $j \in J$, there exist $d_j \in \mathbb{Z}_{\geq 0}$ and $c_j \in \mathbf{k}^{\times}$ such that $\mathfrak{p}_i|_{M_j} = c_j z_j^{d_j}$. Since $\mathfrak{p}_i|_{M_{\mu_1} \circ \dots \circ M_{\mu_m}} = (\mathfrak{p}_i|_{M_{\mu_1}}) \circ \dots \circ (\mathfrak{p}_i|_{M_{\mu_m}}) = \prod_{k=1}^m c_{\mu_k} (x_k^{\mathcal{D}})^{d_{\mu_k}}$, we obtain

$$\begin{aligned} \mathfrak{p}_i|_{N_{\mathfrak{F}}} &= \sum_{\mu \in J^{\gamma}} \left(\mathfrak{p}_i|_{M_{\mu_1} \circ \dots \circ M_{\mu_m}} \right) \otimes_{R^{\mathcal{D}}(\gamma)} N \\ &= \sum_{\mu \in J^{\gamma}} (M_{\mu_1} \circ \dots \circ M_{\mu_m}) \otimes_{R^{\mathcal{D}}(\gamma)} \left(e(\mu) c(x_1^{\mathcal{D}})^{d_{\mu_1}} \dots (x_m^{\mathcal{D}})^{d_{\mu_m}} \right) \Big|_N \\ &= \sum_{\mu \in J^{\gamma}} (M_{\mu_1} \circ \dots \circ M_{\mu_m}) \otimes_{R^{\mathcal{D}}(\gamma)} \left(c e(\mu) \prod_{j \in J} \left(\prod_{k \in [1, m], \mu_k = j} (x_k^{\mathcal{D}})^{d_j} \right) \right) \Big|_N \\ &= \Delta^{\mathcal{D}}(\gamma) \otimes_{R^{\mathcal{D}}(\gamma)} \left(c \prod_{j \in J} \mathfrak{p}_j^{d_j} \right) \Big|_N \end{aligned}$$

with $c = \prod_{k=1}^m c_{\mu_k}$ which does not depend on $\mu \in J^{\gamma}$. Therefore, condition (b) of Definition 2.2 holds.

(ii) (c) immediately follows from (a) and the epimorphism

$$\mathfrak{F}^{\mathcal{D}}(M) \circ \mathfrak{F}^{\mathcal{D}}(N) \twoheadrightarrow \mathfrak{F}^{\mathcal{D}}(M \diamond N)$$

because $M \diamond N$ is simple. □

5. EXAMPLES

Let \mathfrak{g} be a Kac-Moody Lie algebra associated with a Cartan matrix A of finite type. Suppose that

$$(5.1) \quad \begin{cases} \text{(a)} \ \{\beta_j\}_{j \in J} \text{ is a family of elements of } \mathbf{Q}_+, \text{ which is linearly independent} \\ \quad \text{in } \mathbf{Q}, \\ \text{(b)} \ \beta_j - \beta_k \notin \Phi \text{ for any } j, k \in J, \text{ where } \Phi \text{ is the set of roots of } \mathfrak{g}. \end{cases}$$

Let $\bar{\mathfrak{g}}$ be the Lie subalgebra of \mathfrak{g} generated by the root vectors of weight β_j and $-\beta_j$ (cf. [17, Th. 1.1]). Then $\bar{\mathfrak{g}}$ is a Kac-Moody Lie algebra associated to

$$(5.2) \quad \bar{\mathbf{A}} := (\bar{a}_{j,k})_{j,k \in J} \quad \text{with } \bar{a}_{j,k} := 2(\beta_j, \beta_k) / (\beta_j, \beta_j).$$

We have an injective algebra homomorphism

$$(5.3) \quad U^-(\bar{\mathfrak{g}}) \hookrightarrow U^-(\mathfrak{g}).$$

Choosing a convex order of the set Φ_+ of positive roots, let $(\Delta(\beta_j), \mathbf{z}_j)$ be the affinization of $L(\beta_j)$ given in Proposition 3.5. Then, we have the duality datum

$$\mathcal{D} := \{\Delta(\beta_j), \mathbf{z}_j, \mathbf{R}_{k,l}\}_{j,k,l \in J}.$$

Let $\mathfrak{g}^{\mathcal{D}}$ be the Kac-Moody Lie algebra associated with $\mathbf{A}^{\mathcal{D}}$. Suppose that $\mathbf{A}^{\mathcal{D}}$ is of finite type. Then, the functor $\mathfrak{F}^{\mathcal{D}}$ is exact, and gives a $\mathbb{Z}[q^{\pm 1}]$ -algebra homomorphism:

$$[R^{\mathcal{D}}\text{-gmod}] \longrightarrow [R\text{-gmod}]$$

which gives the $\mathbb{Z}[q^{\pm 1/2}]$ -algebra homomorphism (see Corollary 1.6):

$$(5.4) \quad A_q((\mathfrak{g}^{\mathcal{D}})^+)_c \rightarrow \mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(\mathfrak{g}^+).$$

sending f_j to the dual PBW generator $E^*(\beta_j)$ corresponding to $[\Delta(\beta_j)]$. Here \mathbf{c} is the bilinear form on $\mathbf{Q}^{\mathcal{D}}$ given by $\mathbf{c}(\alpha_j^{\mathcal{D}}, \alpha_k^{\mathcal{D}}) = \frac{1}{2}(\deg \mathbf{R}_{k,j} - \deg \mathbf{R}_{j,k})$.

By applying the exact functor $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}[q^{\pm 1/2}]} \bullet$ to (5.4), we obtain a $\mathbb{Q}(q^{1/2})$ -algebra homomorphism

$$(5.5) \quad U_q^-(\mathfrak{g}^{\mathcal{D}})_c \longrightarrow \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}(q)} U_q^-(\mathfrak{g}).$$

Set $c_\beta := (E^*(\beta), E^*(\beta))^{-1}$. Then $E(\beta) = c_\beta E^*(\beta)$ is the PBW vector corresponding to $\beta \in \Phi_+$. Let ψ be the algebra automorphism of $U_q^-(\mathfrak{g}^{\mathcal{D}})_c$ sending f_j to $c_{\beta_j} f_j$. Then the composition

$$U_q^-(\mathfrak{g}^{\mathcal{D}})_c \xrightarrow[\psi]{\sim} U_q^-(\mathfrak{g}^{\mathcal{D}})_c \longrightarrow \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}(q)} U_q^-(\mathfrak{g})$$

sends f_j to $E(\beta_j)$. Since $\deg z_M = (\beta_j, \beta_j)$ by Theorem 3.3, the above homomorphism sends the divided power $f_j^{(m)}$ to the divided power $E(\beta_j)^{(m)}$. Moreover, the $f_j^{(m)}$'s generate the \mathbf{A} -algebra $U_{\mathbf{A}}^-(\mathfrak{g}^{\mathcal{D}})_c$, and the $E(\beta_j)^{(m)}$'s are contained in $U_{\mathbf{A}}^-(\mathfrak{g})$. Hence we obtain the algebra homomorphism

$$(5.6) \quad U_{\mathbf{A}}^-(\mathfrak{g}^{\mathcal{D}})_c \longrightarrow \mathbb{Q}[q^{\pm 1/2}] \otimes_{\mathbb{Q}[q^{\pm 1}]} U_{\mathbf{A}}^-(\mathfrak{g}).$$

Taking the classical limit $q^{1/2} = 1$, we obtain the induced algebra homomorphism

$$(5.7) \quad U^-(\mathfrak{g}^{\mathcal{D}}) \longrightarrow U^-(\mathfrak{g})$$

sending f_j to the root vector corresponding to $-\beta_j$ for $j \in J$.

Proposition 5.1. *If $A^{\mathcal{D}} = \overline{A}$, then the morphism induced by $\mathfrak{F}^{\mathcal{D}}$*

$$[R^{\mathcal{D}}\text{-gmod}] \longrightarrow [R\text{-gmod}]$$

is injective. In particular $\mathfrak{F}^{\mathcal{D}}$ sends the simple $R^{\mathcal{D}}$ -modules to simple R -modules.

In such a case, the functor $\mathfrak{F}^{\mathcal{D}}$ categorifies the homomorphism (5.3).

Proof. By the condition, we have $U^-(\mathfrak{g}^{\mathcal{D}}) \simeq U^-(\overline{\mathfrak{g}})$. Hence the map (5.7) is injective, which implies that (5.6) is injective. Hence (5.5) as well as (5.4) is injective. \square

Let us give several examples of such duality data.

Example 5.2. Let $I = \{1, 2, \dots, \ell\}$ and A the Cartan matrix of type A_{ℓ} . Hence $(\alpha_i, \alpha_j) = 2\delta(i = j) - \delta(|i - j| = 1)$ for $i, j \in I$. Let R be the quiver Hecke algebra associated with A and the parameter $\mathcal{Q}_{i,j}(u, v)$ defined as follows: for $i, j \in I$ with $i < j$,

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} u - v & \text{if } j = i + 1, \\ 1 & \text{otherwise.} \end{cases}$$

Let $J = \{1, 2, \dots, \ell\}$ and $\beta_1 := \alpha_1 + \alpha_2$, $\beta_j := \alpha_j$ for $j \in J \setminus \{1\}$. Note that the β_j 's do not satisfy condition (5.1) (b). We put

$$\Delta(\beta_1) := L(1, 2)_{z_1}, \quad \Delta(\beta_j) := L(j)_{z_j} \quad (j \in J \setminus \{1\}),$$

where $L(1, 2) := \mathbf{k}v$ is the 1-dimensional $R(\beta_1)$ -module with the actions

$$e(\nu)v = \delta_{\nu, (1, 2)}v, \quad x_1v = x_2v = \tau_1v = 0 \quad \text{for } \nu \in I^{\alpha_1 + \alpha_2}.$$

Note that $\deg(z_j) = 2$ for $j \in J$ and the $\Delta(\beta_j)$'s are root modules. We set $R_{j,k} = R_{\Delta(\beta_j), \Delta(\beta_k)}^{\text{norm}}$. By direct computations, the R -matrix $R_{\Delta(\beta_j), \Delta(\beta_k)}$ ($j \neq k$) is given as follows: for $u \otimes v \in \Delta(\beta_j) \otimes \Delta(\beta_k)$,

$$R_{\Delta(\beta_j), \Delta(\beta_k)}(u \otimes v) = \begin{cases} (\tau_2\tau_1(z_2 - z_1) + \tau_1)(v \otimes u) & \text{if } j = 1 \text{ and } k = 2, \\ \tau_2\tau_1(v \otimes u) & \text{if } j = 1 \text{ and } k > 2, \\ \tau_1\tau_2(z_1 - z_2)(v \otimes u) & \text{if } j = 2 \text{ and } k = 1, \\ \tau_1\tau_2(v \otimes u) & \text{if } j > 2 \text{ and } k = 1, \\ \tau_1(v \otimes u) & \text{otherwise,} \end{cases}$$

which yields

$$R_{j,k} = \begin{cases} (z_1 - z_2)^{-1} R_{\Delta(\beta_j), \Delta(\beta_k)} & \text{if } j = 2 \text{ and } k = 1, \\ R_{\Delta(\beta_j), \Delta(\beta_k)} & \text{otherwise,} \end{cases}$$

and

$$\deg(R_{j,k}) = \begin{cases} 1 & \text{if } |j-k| = 1 \text{ and } (j,k) \neq (2,1), \\ 1 & (j,k) = (1,3), (3,1), \\ -1 & \text{if } (j,k) = (2,1), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$A^{\mathcal{D}} = \begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

which is of type D_ℓ , i.e., the quiver Hecke algebra $R^{\mathcal{D}}$ is of type D_ℓ . Note that $\deg(e(1,2)\tau_1^{\mathcal{D}}) = 1$ and $\deg(e(2,1)\tau_1^{\mathcal{D}}) = -1$ (see Definition 1.4). By Theorem 4.4, we have a functor $\mathfrak{F}^{\mathcal{D}}$ between quiver Hecke algebras of type D_ℓ and A_ℓ such that

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j)) \simeq L(\beta_j) \quad \text{for } j \in J.$$

Let us consider the $R^{\mathcal{D}}$ -module $L^{\mathcal{D}}(1,3) := L^{\mathcal{D}}(1) \diamond L^{\mathcal{D}}(3)$ and the one-dimensional R -module $L(1,2,3) := L(1,2) \diamond L(3)$. Applying the functor $\mathfrak{F}^{\mathcal{D}}$ to the exact sequence

$$0 \rightarrow L^{\mathcal{D}}(1,3) \rightarrow L^{\mathcal{D}}(3) \circ L^{\mathcal{D}}(1) \rightarrow L^{\mathcal{D}}(1) \circ L^{\mathcal{D}}(3) \rightarrow L^{\mathcal{D}}(1,3) \rightarrow 0,$$

we have

$$0 \rightarrow \mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(1,3)) \rightarrow L(3) \circ L(1,2) \rightarrow L(1,2) \circ L(3) \rightarrow \mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(1,3)) \rightarrow 0.$$

Since $\mathfrak{F}^{\mathcal{D}}$ sends a simple module to a simple module or zero by Theorem 4.4 and $L(3) \circ L(1,2)$ is not isomorphic to $L(1,2) \circ L(3)$, we have

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(1,3)) \simeq L(1,2,3).$$

Set $L^{\mathcal{D}}(1,3,2) \simeq L^{\mathcal{D}}(1,3) \diamond L^{\mathcal{D}}(2)$, which is one-dimensional. It is isomorphic to the image of the composition of

$$\begin{aligned} L^{\mathcal{D}}(1) \circ L^{\mathcal{D}}(3) \circ L^{\mathcal{D}}(2) &\longrightarrow L^{\mathcal{D}}(1) \circ L^{\mathcal{D}}(2) \circ L^{\mathcal{D}}(3) \longrightarrow L^{\mathcal{D}}(2) \circ L^{\mathcal{D}}(1) \circ L^{\mathcal{D}}(3) \\ &\longrightarrow L^{\mathcal{D}}(2) \circ L^{\mathcal{D}}(1,3). \end{aligned}$$

By applying $\mathfrak{F}^{\mathcal{D}}$, we obtain the diagram

$$(5.8) \quad \begin{array}{c} L(1, 2) \circ L(3) \circ L(2) \xrightarrow{f_1} L(1, 2) \circ L(2) \circ L(3) \xrightarrow{f_2} L(2) \circ L(1, 2) \circ L(3) \\ \downarrow f_3 \\ L(2) \circ L(1, 2, 3). \end{array}$$

Hence $\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(1, 3, 2))$ is isomorphic to the image of $f_3 f_2 f_1$. Let $u_{1,2}$, u_2 , and u_3 be the generator of $L(1, 2)$, $L(2)$ and $L(3)$, respectively. Then we have

$$\begin{aligned} f_1(u_{1,2} \otimes u_3 \otimes u_2) &= \tau_3(u_{1,2} \otimes u_2 \otimes u_3), \\ f_2(u_{1,2} \otimes u_2 \otimes u_3) &= \tau_1(u_2 \otimes u_{1,2} \otimes u_3). \end{aligned}$$

Therefore, we obtain

$$f_2 f_1(u_{1,2} \otimes u_3 \otimes u_2) = \tau_3 \tau_1(u_2 \otimes u_{1,2} \otimes u_3) = \tau_1 \tau_3(u_2 \otimes u_{1,2} \otimes u_3),$$

which is killed by f_3 . Thus $f_3 f_2 f_1 = 0$, and hence we conclude

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(1, 3, 2)) \simeq 0.$$

Therefore, $\mathfrak{F}^{\mathcal{D}}$ can send simple modules to zero in this example.

6. FURTHER EXAMPLES FOR NON-SYMMETRIC TYPES

Let $\beta \in \mathbf{Q}_+$ and let $(\mathbf{M}, z_{\mathbf{M}})$ be an affinization of a real simple $R(\beta)$ -module \bar{M} . We set $J = \{0\}$, $\beta_0 = \beta$, $\mathbf{M}_0 = \mathbf{M}$. Then

$$\mathcal{D} = \{\mathbf{M}_0, z_{\mathbf{M}}, R_{\mathbf{M}, \mathbf{M}}^{\text{norm}}\}$$

is a duality datum. Then the corresponding simple root $\alpha_0^{\mathcal{D}}$ satisfies $(\alpha_0^{\mathcal{D}}, \alpha_0^{\mathcal{D}}) = \deg z_{\mathbf{M}}$. Let $(\mathbf{K}(0^n), z_{\mathbf{K}(0^n)})$ be the affinization of the simple $R^{\mathcal{D}}(n\alpha_0^{\mathcal{D}})$ -module $L(\alpha_0^{\mathcal{D}})^{\circ n}$ given in Example 2.18.

Now $\mathbf{M}^{\circ n} := \underbrace{\mathbf{M} \circ \cdots \circ \mathbf{M}}_{n \text{ times}}$ has a structure of $(R(n\beta), R^{\mathcal{D}}(n\alpha_0^{\mathcal{D}}))$ -bimodule. We set

$$C_n(\mathbf{M}) = \mathbf{M}^{\circ n} \otimes_{R^{\mathcal{D}}(n\alpha_0^{\mathcal{D}})} \mathbf{K}(0^n) \simeq \mathfrak{F}^{\mathcal{D}}(\mathbf{K}(0^n)).$$

Then $z_{\mathbf{K}(0^n)} \in \text{END}(\mathbf{K}(0^n))$ induces an endomorphism $z_{C_n(\mathbf{M})} \in \text{END}(C_n(\mathbf{M}))_{n \deg z_{\mathbf{M}}}$. By Theorem 4.4 (ii) (b), we obtain the following lemma.

Lemma 6.1. *$(C_n(\mathbf{M}), z_{C_n(\mathbf{M})})$ is an affinization of the real simple module $\bar{M}^{\circ n}$.*

For example,

$$(6.1) \quad C_2(M) = \frac{M \circ M}{(z_M \circ M + M \circ z_M)(M \circ M) + r(M \circ M)},$$

where r is the endomorphism given in (4.3). $z_{C_2(M)}$ is the endomorphism induced by $z_M \circ z_M$.

Let $M \in R(\beta)\text{-gmod}$ and $N \in R(\beta')\text{-gmod}$ be real simple modules. Suppose that $R(\beta)$ and $R(\beta')$ are symmetric, and

$$R_{N_{z'}, M_t}^{\text{norm}} R_{M_t, N_{z'}}^{\text{norm}} = c(t - z')^p \in \text{End}_{R(\beta+\beta')}(M_t \circ N_{z'})$$

for some $c \in \mathbf{k}^\times$ and $p \in \mathbb{Z}_{\geq 0}$. Set

$$\begin{aligned} R_1 &:= (R_{M_{t_1}, N_{z'}}^{\text{norm}} \circ 1)(1 \circ R_{M_{t_2}, N_{z'}}^{\text{norm}}) \in \text{Hom}_{R(2\beta+\beta')}(M_{t_1} \circ M_{t_2} \circ N_{z'}, N_{z'} \circ M_{t_1} \circ M_{t_2}), \\ R_2 &:= (1 \circ R_{N_{z'}, M_{t_2}}^{\text{norm}})(R_{N_{z'}, M_{t_1}}^{\text{norm}} \circ 1) \in \text{Hom}_{R(2\beta+\beta')}(N_{z'} \circ M_{t_1} \circ M_{t_2}, M_{t_1} \circ M_{t_2} \circ N_{z'}). \end{aligned}$$

Setting $t_1 + t_2 = 0$ and $t_1 t_2 = \widehat{z} := z_{C_2(M)}$, we regard R_1 and R_2 as homomorphisms in $\text{Hom}_{R(2\beta+\beta')}(C_2(M_z) \circ N_{z'}, N_{z'} \circ C_2(M_z))$ and $\text{Hom}_{R(2\beta+\beta')}(N_{z'} \circ C_2(M_z), C_2(M_z) \circ N_{z'})$, respectively. Then, we have

$$\begin{aligned} R_2 R_1 &= c^2(t_1 - z')^p (t_2 - z')^p \\ (6.2) \quad &= c^2(t_1 t_2 - (t_1 + t_2)z' + z'^2)^p \\ &= c^2(\widehat{z} + z'^2)^p \end{aligned}$$

in $\text{End}_{R(2\beta+\beta')}(C_2(M_z) \circ N_{z'})$.

Using (6.2), one can construct functors $\mathfrak{F}^{\mathcal{D}}$ between symmetric and non-symmetric quiver Hecke algebras. In particular, the functor from type C_ℓ (resp. $C_\ell^{(1)}$, $A_{2\ell-1}^{(2)}$) to type A_ℓ (resp. $A_{\ell+1}$, $D_{\ell+1}$) can be constructed. We give such constructions in the following examples.

Example 6.2. We take I , A , and $\mathcal{Q}_{i,j}(u, v)$ given in Example 5.2. In particular, \mathfrak{g} is of type A_ℓ .

Let $J = \{1, 2, \dots, \ell\}$ and

$$\beta_1 = 2\alpha_1, \quad \beta_j = \alpha_j \text{ for } j \in J \setminus \{1\}.$$

Let us denote

$$M_1 = K(1^2), \quad M_j = L(j)_{z_j} \quad (j \in J \setminus \{1\}),$$

and $z_1 := z_{K(1^2)}$. Then $\deg z_1 = 4$ and $\deg z_j = 2$ for $j \neq 1$. Note that

$$R_{L(j)_z, L(k)_w}^{\text{norm}} = R_{L(j)_z, L(k)_w}.$$

We put $R_{j,k} := R_{M_j, M_k}$. It follows from (6.2) that, for $j, k \in J$ with $j < k$,

$$(6.3) \quad R_{k,j} R_{j,k} = \begin{cases} z_1 \circ 1 + 1 \circ z_2^2 & \text{if } (j, k) = (1, 2), \\ z_j \circ 1 - 1 \circ z_k & \text{if } k = j + 1 \text{ and } (j, k) \neq (1, 2), \\ 1 & \text{otherwise.} \end{cases}$$

We now set

$$\mathcal{D} = \{M_j, z_j, R_{j,k}\}_{j,k \in J}.$$

Then \mathcal{D} is a dual datum, and (6.3) implies that

$$A^{\mathcal{D}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -2 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

which is of type C_ℓ . Therefore, we have the quiver Hecke algebra $R^{\mathcal{D}}$ of type C_ℓ and the functor $\mathfrak{F}^{\mathcal{D}}$ from the category of modules over quiver Hecke algebras of type C_ℓ to that of type A_ℓ sending

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j)) \simeq \begin{cases} L(1) \circ L(1) & \text{if } j = 1, \\ L(j) & \text{otherwise.} \end{cases}$$

We conjecture that $\mathfrak{F}^{\mathcal{D}}$ sends simple to simple.

We make the following examples for type B_ℓ by constructing affinizations directly.

Example 6.3. Let $I = \{1, 2, \dots, \ell\}$ and A the Cartan matrix of type B_ℓ :

$$A = \begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix},$$

and $(\alpha_i, \alpha_j) = 2\delta(i = j = 1) + 4\delta(i = j \neq 1) - 2\delta(|i - j| = 1)$ for $i, j \in I$.

Let R be the quiver Hecke algebra associated with \mathbf{A} and the parameter $\mathcal{Q}_{i,j}(u, v)$ defined as follows: for $i, j \in I$ such that $i < j$,

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} u^2 - v & \text{if } (i, j) = (1, 2), \\ u - v & \text{if } j = i + 1 \text{ and } (i, j) \neq (1, 2), \\ 1 & \text{otherwise.} \end{cases}$$

Let $J = \{1, 2, \dots, \ell - 1\}$ and

$$\beta_1 = \alpha_1 + \alpha_2, \quad \beta_j = \alpha_{j+1} \quad (j \in J \setminus \{1\}).$$

Note that $(\beta_1, \beta_1) = 2$ and $(\beta_j, \beta_j) = 4$ for $j \neq 1$. We put

$$\Delta(\beta_1) = L(1, 2)_{\mathbf{z}_1}, \quad \Delta(\beta_j) = L(j+1)_{\mathbf{z}_j} \quad (j \neq 1),$$

where the $R(\beta_1)$ -module $L(1, 2)_{\mathbf{z}_1} := \mathbf{k}[\mathbf{z}_1]v$ is defined by

$$e(\nu)v = \delta_{\nu, (1,2)}v, \quad x_j v = \mathbf{z}_1^{(\alpha_j, \alpha_j)/2}v, \quad \tau_1 v = 0.$$

Note that $\Delta(\beta_j)$'s are root modules and $\deg(\mathbf{z}_j)$ is 2 or 4 according that $j = 1$ or not. For $j, k \in J$ with $j \neq k$, we define

$$\mathbf{R}_{j,k} := R_{\Delta(\beta_j), \Delta(\beta_k)} \in \text{Hom}_{R(\beta_j + \beta_k)}(\Delta(\beta_j) \circ \Delta(\beta_k), \Delta(\beta_k) \circ \Delta(\beta_j)),$$

that is,

$$\mathbf{R}_{j,k}(p \otimes q) = \begin{cases} \tau_2 \tau_1(q \otimes p) & \text{if } j = 1, \\ \tau_1 \tau_2(q \otimes p) & \text{if } k = 1, \\ \tau_1(q \otimes p) & \text{otherwise} \end{cases}$$

for $p \otimes q \in \Delta(\beta_j) \otimes_{\mathbf{k}} \Delta(\beta_k)$. For $j, k \in J$ with $j < k$, we have

$$\mathbf{R}_{k,j} \mathbf{R}_{j,k} = \begin{cases} \mathbf{z}_j^2 - \mathbf{z}_k & \text{if } (j, k) = (1, 2), \\ \mathbf{z}_j - \mathbf{z}_k & \text{if } k = j + 1 \text{ and } (j, k) \neq (1, 2), \\ 1 & \text{otherwise.} \end{cases}$$

Then we have the duality datum $\mathcal{D} = \{\Delta(\beta_j), \mathbf{z}_j, \mathbf{R}_{j,k}\}_{j,k \in J}$ and

$$\mathbf{A}^{\mathcal{D}} = \begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

which is of type $B_{\ell-1}$. Therefore, we have the functor $\mathfrak{F}^{\mathcal{D}}$ from the category of modules over quiver Hecke algebra of type $B_{\ell-1}$ to that of type B_{ℓ} sending

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j)) \simeq \begin{cases} L(1, 2) & \text{if } j = 1, \\ L(j+1) & \text{otherwise,} \end{cases}$$

where $L(1, 2) = L(1, 2)_{\mathbf{z}_1}/\mathbf{z}_1 L(1, 2)_{\mathbf{z}_1}$.

It is easy to check that $\{\beta_1, \dots, \beta_{\ell-1}\}$ satisfies (5.1) and $\mathbf{A}^{\mathcal{D}}$ is equal to the matrix $\overline{\mathbf{A}}$ defined by (5.2). Thus, Proposition 5.1 implies that the functor $\mathfrak{F}^{\mathcal{D}}$ categorifies the injective homomorphism $U^-(\mathfrak{g}^{\mathcal{D}}) \simeq U^-(\overline{\mathfrak{g}}) \rightarrow U^-(\mathfrak{g})$ and $\mathfrak{F}^{\mathcal{D}}$ sends the simple modules to simple modules. By Theorem 4.4, for a simple $R^{\mathcal{D}}$ -module N ,

$$\mathfrak{F}^{\mathcal{D}}(N \diamond L^{\mathcal{D}}(j)) \simeq \begin{cases} \mathfrak{F}^{\mathcal{D}}(N) \diamond L(1, 2) & \text{if } j = 1, \\ \mathfrak{F}^{\mathcal{D}}(N) \diamond L(j+1) & \text{otherwise.} \end{cases}$$

Example 6.4. We use the same notations I , \mathbf{A} , and $\mathcal{Q}_{i,j}(u, v)$ as in Example 6.3.

Let $J = \{1, 2, \dots, \ell-1\}$ and

$$\beta_1 = 2\alpha_1 + \alpha_2, \quad \beta_j = \alpha_{j+1} \quad (j \in J \setminus \{1\}).$$

Note that $(\beta_j, \beta_j) = 4$ for all $j \in J$. We define an $R(\beta_1)$ -module structure on $L(1, 1, 2)_{\mathbf{z}_1} := \mathbf{k}[\mathbf{z}_1] \otimes_{\mathbf{k}} (\mathbf{k}u \oplus \mathbf{k}v)$ by

$$\begin{aligned} e(\nu)(a \otimes u) &= \delta_{\nu, (1,1,2)} a \otimes u, & e(\nu)(a \otimes v) &= \delta_{\nu, (1,1,2)} a \otimes v, \\ x_j(a \otimes u) &= \begin{cases} -\mathbf{z}_1 a \otimes v & \text{if } j = 1, \\ \mathbf{z}_1 a \otimes v & \text{if } j = 2, \\ \mathbf{z}_1 a \otimes u & \text{otherwise,} \end{cases} & x_j(a \otimes v) &= \begin{cases} -a \otimes u & \text{if } j = 1, \\ a \otimes u & \text{if } j = 2, \\ \mathbf{z}_1 a \otimes v & \text{otherwise,} \end{cases} \\ \tau_k(a \otimes u) &= \begin{cases} a \otimes v & \text{if } k = 1, \\ 0 & \text{if } k \neq 1, \end{cases} & \tau_k(a \otimes v) &= 0 \quad \text{for any } k. \end{aligned}$$

We put

$$\Delta(\beta_1) = L(1, 1, 2)_{\mathbf{z}_1}, \quad \Delta(\beta_j) = L(j+1)_{\mathbf{z}_j} \quad (j \neq 1).$$

Note that $\Delta(\beta_j)$'s are root modules and $\deg(\mathbf{z}_j) = 4$ for $j \in J$. For $j, k \in J$ with $j \neq k$ and $p \otimes q \in \Delta(\beta_j) \otimes_{\mathbf{k}} \Delta(\beta_k)$, we define

$$R_{j,k} := R_{\Delta(\beta_j), \Delta(\beta_k)} \in \text{Hom}_{R(\beta_j + \beta_k)}(\Delta(\beta_j) \circ \Delta(\beta_k), \Delta(\beta_k) \circ \Delta(\beta_j)).$$

Then,

$$R_{k,j} R_{j,k} = \begin{cases} \mathbf{z}_j - \mathbf{z}_k & \text{if } k = j+1, \\ 1 & \text{otherwise,} \end{cases}$$

for $j, k \in J$ with $j < k$.

Then, we have a duality datum $\mathcal{D} = \{\Delta(\beta_j), z_j, R_{j,k}\}_{j,k \in J}$ and

$$A^{\mathcal{D}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

is of type $A_{\ell-1}$. Therefore, we have the quiver Hecke algebra $R^{\mathcal{D}}$ of type $A_{\ell-1}$ and the functor $\mathfrak{F}^{\mathcal{D}}$ between quiver Hecke algebras of type $A_{\ell-1}$ and B_{ℓ} . Moreover,

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j)) \simeq \begin{cases} L(1, 1, 2) & \text{if } j = 1, \\ L(j+1) & \text{otherwise,} \end{cases}$$

where $L(1, 1, 2) = L(1, 1, 2)_{z_1}/z_1 L(1, 1, 2)_{z_1}$.

One can easily show that $\{\beta_1, \dots, \beta_{\ell-1}\}$ satisfies (5.1) and $A^{\mathcal{D}}$ is equal to the matrix \bar{A} defined by (5.2). Thus, Proposition 5.1 implies that the functor $\mathfrak{F}^{\mathcal{D}}$ categorifies the injective homomorphism $U^-(\bar{\mathfrak{g}}) \rightarrow U^-(\mathfrak{g})$ and $\mathfrak{F}^{\mathcal{D}}$ preserves simple modules. We have

$$\mathfrak{F}^{\mathcal{D}}(N \diamond L^{\mathcal{D}}(j)) \simeq \begin{cases} \mathfrak{F}^{\mathcal{D}}(N) \diamond L(1, 1, 2) & \text{if } j = 1, \\ \mathfrak{F}^{\mathcal{D}}(N) \diamond L(j+1) & \text{otherwise,} \end{cases}$$

for a simple $R^{\mathcal{D}}$ -module N by Theorem 4.4.

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